MONTHLY NOTICES

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Dr. J. H. JEANS, PRESIDENT, in the Chair.

Thomas Davies, B.A., B.Sc., 82 Warley Road, Langley, near Birmingham; and

Rev. James William St. Clare Hill, M.A., Principal of the Royal School for the Blind, Leatherhead, Surrey,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Robert Morrison Abraham, Luthers, Sewardstone, London, E. 4 (proposed by E. A. Reeves);

William Reginald Dean, M.A., Fellow of Trinity College, Cambridge, and Assistant Professor of Mathematics, Imperial College of Science and Technology, South Kensington, London, S.W. 7 (proposed by E. A. Milne);

Joseph Cecil Maby, Marl Hill, near Cheltenham, Gloucestershire (proposed by T. E. R. Phillips); and

Edward Wilfred Taylor, 10 Telford Terrace, York (proposed by T. E. R. Phillips).

Eighty-three presents were announced as having been received since the last meeting, including, amongst others :—

Bordeaux Observatory, Carte photographique du Ciel, zones +14°, +12° (9 charts).

Toulouse Observatory, Carte photographique du Ciel, zones +5°, +7°, +9° (16 charts).

Washington, United States Naval Observatory, Publications, Appendix to vol. x., Total Solar Eclipses, 1905, 1918, 1923.

The Spectrum of φ Persei (Type Bope).

By William J. S. Lockyer, M.A., Ph.D. (Plates 16, 17.)

In a communication to the Society in April of last year, entitled "Recent Observations of some Bright-hydrogen-line Stars," * I pointed out that the photographs of the spectra of φ Persei (H.D.C. 10516. R.A. $1^h\ 37^m{\cdot}4$. Dec. $+50°\ 11'$. Mag. 4·19. Sp. Bope) taken during the years 1923 and 1924 indicated clearly that the bright hydrogen lines underwent many variations.

Since these observations were made, the spectrum of this star has been rephotographed a great many times in order not only to determine any cyclical change in the appearance of the hydrogen lines but to investigate the nature and origin of the other sharp bright lines and also the sharp and hazy dark lines which are peculiar and prominent features of the spectrum.

This communication is limited strictly to the facts of observation, and no attempt is made to explain the origin of the variations. The reason for this is that there are changes going on in the spectrum which take place every few days, and until a greater number of photographs is secured these changes cannot be accurately determined.

The Material Available.

All the photographs which are here discussed were taken by the writer with the 9-inch prismatic camera of the Norman Lockyer Observatory, which has a focal length of 120 inches and a prism of 45° angle, the dispersion being 46·0 mm. between H_β and H_δ.

Exposures of about an hour or more were usually given to secure the requisite density and breadth of spectra, and the plates used were the Ilford "Iso-Zenith," having a speed number of H and D 700.

Four of the photographs, namely numbers 355, 362, 403, and 435, were taken in connection with the first paper on the bright-hydrogen-line stars, and since then 51 more negatives have been secured, thus making the total number of 55 plates for this discussion. The recent photographs were taken between 1924 August 8 and 1925 April 12.

The accompanying table (Table I.) gives a list of the photographs available for the inquiry, showing at the heads of the columns the Norman Lockyer Observatory negative numbers, the year, month, and day, middle time of exposure, and corresponding Julian day. (No change has been made in the Julian day since 1925 January 1.)

Cyclical Change of the Bright Hydrogen Lines.

It should first be pointed out that each of the hydrogen lines in this star consists of a broad absorption band on which are superposed two bright lines symmetrically placed with regard to it and separated from each other by a strong, sharp absorption line. In other words, each

* *Monthly Notices*, **84**, 409.

hydrogen line is termed " double bright " on wide absorption. This can be seen in the reproductions of some of the spectra as illustrated on Plates 16 and 17 accompanying this paper.

The new photographs showed at once that the intensities of the two bright components in any one hydrogen line changed in relation to each other in a gradual and progressive manner. Thus in the case of H_β, for example, sometimes the violet component would be brighter than the red component, then they would gradually become equal in intensity, and eventually the red component would outshine the violet. All the other hydrogen lines conformed exactly to the same behaviour as H_β, only not to such a great degree.

To determine the nature of these changes it was decided to measure the relative intensities of the components of any one hydrogen line in all the photographs.

In the first instance these differences of intensity between the violet and red components were made by eye estimation by the writer. This, however, was not considered very satisfactory as there were so many grades of difference between the two lines, but as it turned out afterwards the estimates were very fair approximations.

It was thought, however, that it might be possible to determine these differences of intensity more accurately by using the graduated wedge under the Hilger micrometer, as has been done at this Observatory in the case of close absorption lines for spectroscopic parallax determinations. An attempt by the writer showed clearly that this wedge method could be very satisfactorily applied to these bright components.

In order to eliminate any possible bias in the measures, the negatives were given by the writer in a promiscuous order to Mr. D. L. Edwards to measure, Mr. Edwards being now very experienced in the use of the wedge-measuring machine by its employment in his measures for the spectroscopic parallaxes of the B-type stars.

It was considered necessary to measure only the components of H_β and H_γ, the latter to serve chiefly as a check on the former.

The wedge method consists in placing the negative under the microscope so that the components of the hydrogen line are in the centre of the field of view. Under the negative is a fixed graduated wedge, and under the wedge is a plane reflector to send a constant light into the microscope. A measure thus consists in moving by the micrometer handle the microscope, negative, and reflector simultaneously until each bright line is made to disappear by some portion of the wedge. A scale gives the reading corresponding to the particular portion of the wedge which makes the line disappear. For each line the mean of ten settings was adopted. When both components of a hydrogen line have been measured, the difference of the readings gives a value for the intensity difference.

The values thus obtained are given in Table I., columns (7) and (8). The plus signs indicate that the red component was brighter than the violet, and the minus signs that the violet was brighter than the red. The question marks indicate that the image of the line in question was not very satisfactory for measurement.

TABLE I.

Relative Intensities of the Components of H_β and H_γ.

Neg. No.	Year.	Month and Day.	Exp. G.M.T.	Julian Day 2420000+.	Phase Days.	Diff. in Int. Red minus Violet Components.	
						H_β.	H_γ.
(1).	(2).	(3).	(4). h m	(5).	(6).	(7).	(8).
355	1923	Sept. 10	11 37	3673·5	58	+2·4	+2·1
362		11	11 35	3674·5	59	+2·2	+2·0
403		Dec. 13	6 20	3767·3	25	0·0	0·0
435	1924	Feb. 14	8 6	3830·3	88	+0·5	+0·2
472		Aug. 8	14 24	4006·6	10	−2·2	−1·8
475		13	12 56	4011·5	15	−1·8	−1·5 ?
483		20	15 1	4018·6	22	−0·9	−0·9
489		Sept. 2	12 56	4031·5	35	+0·3	+0·1 ?
497		Oct. 23	13 0	4082·5	86	+0·9	−0·1
500		Nov. 3	13 16	4093·5	97	−0·5	0·0
509		12	10 52	4102·5	106	−1·0	−0·8
510		16	11 18	4106·5	110	−1·2	−0·9
515		17	10 49	4107·5	111	−1·2 ?	−1·0
522		19	10 47	4109·5	113	−1·5	−1·3
530		Dec. 3	6 37	4123·3	0	−2·3 ?	−2·0
536		5	7 1	4125·3	2	−2·2	−2·0
537		14	5 45	4134·3	11	−2·0	−1·9
543		24	5 42	4144·3	21	−1·7	..
544		26	6 7	4146·3	23	−1·6	−1·4
			G.C.T.				
548	1925	Jan. 4	18 15	4155·3	32	..	−0·3
550		5	18 22	4156·3	33	−0·2	−0·1
555		15	18 27	4166·3	43	+1·2	+0·9
558		23	21 8	4174·4	51	+2·0	+1·8
564		28	21 12	4179·4	56	+2·0	+1·8
568		31	21 7	4182·4	59	+2·2	+2·0
569		Feb. 1	18 51	4183·3	60	+2·5	+2·2
570		1	19 55	4183·3	60	+2·3	+2·0
571		2	19 12	4184·3	61	+2·5	+2·0
575		6	18 52	4188·3	65	+2·4	+2·0
576		6	19 47	4188·3	65	+2·5	+2·0
577		12	19 47	4194·3	71	+2·2	..
578		18	20 59	4200·4	77	+1·8	+1·5
581		20	19 25	4202·3	79	+1·6	+1·2
583		23	19 12	4205·3	82	+1·2	+0·8
584		23	20 45	4205·4	82	+1·3	+1·0

TABLE I.—*continued.*

Neg. No.	Year.	Month and Day.	Exp. G.C.T.	Julian Day 2420000d+.	Phase Days.	Diff. in Int. Red minus Violet Components.	
						H_β.	H_γ.
(1).	(2).	(3).	(4). h m	(5).	(6).	(7).	(8).
585	1925	Feb 27	19 50	4209·3	86	+0·9	+0·4
588		Mar. 2	20 35	4212·4	89	+0·5	0·0
590		8	19 49	4218·3	95	−0·3	0·0
592		9	20 6	4219·3	96	−0·4	0·0
595		11	20 21	4221·3	98	−0·9	−0·4
597		15	19 42	4225·3	102	−1·2	−0·9
598		17	20 3	4227·3	104	−1·2	−0·9
601		18	20 15	4228·3	105	−1·3	−1 0
605		21	20 37	4231·3	108	−1·2	−1·0
606		23	20 2	4233·3	110	−1·4	−1·1
608		27	20 25	4237·3	114	−1·8	−1·3 ?
611		Apr. 10	21 0	4251·3	1	−2·4	−2·0 ?

It will be noticed that the differences in intensity between the components of H_β were always greater than those of H_γ, and the same may be said about H_γ in relation to H_δ, although no actual measures were made. This diminishing difference of intensity between the components of the hydrogen lines seems to be a general feature of all the bright-hydrogen-line stars that have been examined.

In order to show clearly the sequence of changes of the intensities of the bright hydrogen components one set of values, namely that for H_β, has been plotted and is here reproduced in the accompanying figure (fig. 1).

The ordinates represent the values given in column (7) of Table I., and the abscissæ the dates given in columns (2) to (5). At the top of the figure are added the corresponding phases of increase and decrease of positive and negative velocity as explained further on.

The first four values given in Table I. have not been included in this figure, as they correspond to observations made in 1923 and 1924 February and would make the figure too long. They are, however, included in the discussion later.

A smooth curve was first drawn through all the observations, including those made in 1923 and early 1924, and this was found to be approximately symmetrical with a period of about 128 days. A very good series of observations was made about the last maximum, which indicated 1925 February 4 (J.D. 2424186) as the epoch of maximum, and with +2·5 as the difference between the intensities of the components of H_β. On 1923 September 10 (J.D. 2423673) the difference between the intensities of the components of H_β was measured as +2·4, so that this date represents an epoch very close to, if not at, a maximum. The interval between these dates is 513 days, and as four periods lie between them, the length of a period is 128·2 days.

Again on 1924 August 29 (J.D. 2424027) the components of H_β are of equal intensity: at the same phase in the following period the date is 1925 January 6 (J.D. 2424157). The interval between these dates amounts to 130 days.

On 1924 October 29 (J.D. 2424088) the curve cuts the zero line where the intensities of the components of H_β are equal. A similar phase is indicated a period later on 1925 March 6 (J.D. 2424216) which gives the length of a period as 128 days.

The mean of the above values is 128·7 days. This value is of course only approximate. A much more accurate determination of the length of the period could be obtained by using an epoch many years back, and this has been done. It is shown further on that, by utilising the date of a minimum in 1909 as determined by Jordan and a similar phase on 1924 December 3 as shown in fig. 1, the mean value of the period is 126·8 days.

Previous work on the binary nature of this star has all been accomplished by the measurements of the displacements of the hydrogen lines with a slit spectroscope and a comparison spectrum.

Campbell and Reece,* in the year 1902, first announced its binary nature from radial velocity measurements of the narrow absorption line at H_γ.

In 1910 J. B. Cannon † published a paper on the orbit of this star. In this investigation he used 120 plates. As he states, "The absorption bands were used entirely in the work, an attempt to obtain measures on the bright lines either directly or by means of a spectro-comparator having been found useless." The actual lines he worked with were H_β, H_γ, H_δ, H_ϵ, the helium line $\lambda 4471$ and K ($\lambda 3933{\cdot}8$). The period he found to be 126·5 days.

In the same year Ludendorff ‡ published an orbit determination deducing a period of 126·6 days from the absorption line at H_γ.

F. C. Jordan, in 1913, published § some spectrographic observations of this star, but in this investigation he made no determination of the period. This paper, it may be noted, contains a brief historical summary of previous work on this star and numerous references to the literature of this subject.

It will be seen, therefore, that the period deduced from the curve formed to satisfy the variations in intensities between the components of the hydrogen lines agrees very well with that given by the determination of the velocities in the line of sight by the slit spectroscope.

To interpret in terms of orbital velocity the changes in the intensity of the components of the hydrogen lines, it should be stated that it is assumed that the broad emission line or band only appears double because of the presence of a sharp and strong absorption line blotting out its central portion. If this absorption line is situated symmetrically on the broad emission band, both of the bright components

* *Lick Obs. Bull.*, **1**, No. 20, 158.
† *Journ. Roy. Astr. Soc. Canada*, **4**, 195, 1910.
‡ *Astr. Nach.*, 4442.
§ *Pub. All. Obs. Pittsburgh*, **3**, 31, No. 5, 1913.

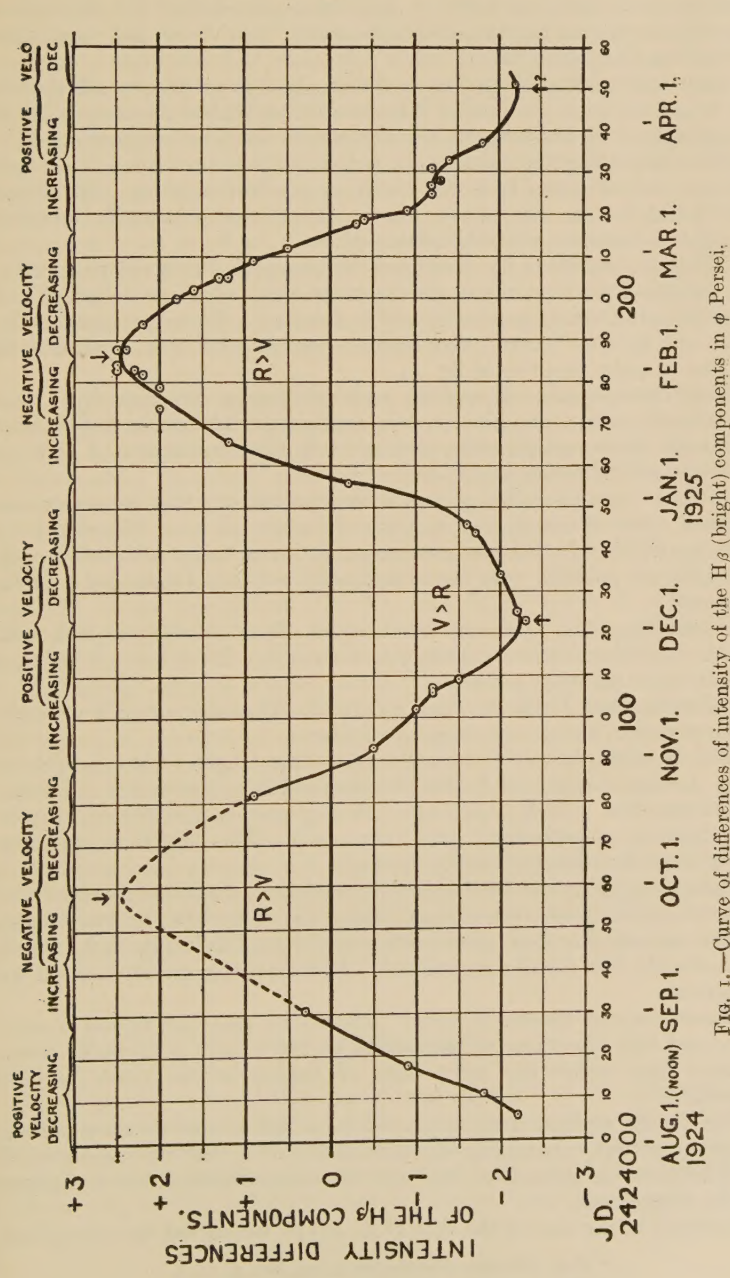


FIG. 1.—Curve of differences of intensity of the Hβ (bright) components in φ Persei.

being then of equal intensity, it may be assumed that the absorption line is in its normal position and indicates a " zero " velocity in relation to the broad emission line.

Any unsymmetrical position of the absorption line in relation to the broad emission line would therefore be equivalent to motion in the line of sight. If, for instance, the absorption line were unsymmetrical—that is, shifted slightly towards the red end of the spectrum—this would indicate motion away from the earth, or positive velocity. If, on the other hand, it were shifted towards the violet, motion towards the earth or negative velocity would be indicated.

The curve shown in fig. 1 can now be interpreted as a velocity curve. The points of maximum are epochs of greatest negative velocity, and those of minimum of greatest positive velocity. Those portions of the curve cut by the " zero " line indicate epochs of " zero " velocity in the line of sight (see top of fig. 1).

That these epochs of maxima and minima, as shown in fig. 1, and determined by line intensities, are in conformity with those determined by the slit spectroscopic observations of motion in the line of sight is shown by the following statement.

Jordan * computes his phases with a period of 126·5 days from an assumed time of maximum positive velocity on 1909 December 29 (J.D. 2418670). In the present investigation a fairly accurate epoch of maximum positive velocity is indicated on 1924 December 3 (J.D. 2424123).

Calculating this last-mentioned epoch from Jordan's epoch and period the Julian date 2424109·5 is obtained. This is short by 13·5 days after a lapse of 43 periods.

Assuming that Jordan's epoch is correct, either the period is variable or the period of 126·5 days is not accurate.

It is possible, however, to redetermine the length of the period by using Jordan's epoch and that determined by the writer, namely, 1924 December 3 (J.D. 2424123). This gives an interval of time of 5453 days, or 43 periods of 126·8 days each. This period is 0·3 days longer than that determined by Jordan.

Calculating now the later epoch of maximum positive velocity from Jordan's epoch (given above) and using the new value of 126·8 days for the period, the new epoch falls on Julian Day 2424122·4, which is practically the same date, namely, J.D. 2424123, as determined by the writer.

There is every reason to believe, therefore, that the period is constant, and that the value of the period of 126·8 days is probably more accurate than either the 126·5 days of Jordan or the 126·6 days of Ludendorff.

The wedge method here employed is, so far as can be ascertained, unique in actually deducing the periodicity of a star by measures of the differences in intensities between the components of the hydrogen lines in a spectrum.

Curtiss,† in the case of the star H.D. 20336, estimated the intensities

* *Pub. All. Obs. Pittsburgh*, **3**, 35, No. 5, 1913.
† *Pub. Astr. Obs. Michigan*, **2**, 40, 1916.

of each of the bright components of the hydrogen line, not against each other but each with reference to the neighbouring continuous spectrum. Only the fact that the observations were not continued over a sufficiently long period of time prevented him from determining the star's period by this method.

It has long been known that the bright components of the hydrogen lines in φ Persei underwent changes synchronising with the orbital period, although no actual measures seem to have been made.

Thus Ludendorff,* in 1910, pointed out that the appearance of the H_γ line was most peculiar, showing regular periodic variations.

Merrill,† in 1913, writing about the spectrum of φ Persei, stated :—

" H lines variable in appearance in same cycle as orbital period."

Jordan,‡ in the same year, refers in a general manner to the changes which take place in the spectrum at different parts of its orbit.

In order to form a normal curve, all the observations recorded in Table I. have been reduced to one period, using as a basis the length of period of 126·8 days and the date 1924 December 3 (J.D. 2424123) as the epoch of maximum positive velocity or zero day. The phase of each observation is given in column (6) of Table I.

These points are all plotted in fig. 2 (upper portion) and a smooth curve drawn through them. The points above the " zero " line represent the measures when the red component of the H_β line was brighter than the violet component (R>V), and those below it when the violet was brighter than the red component (V>R). It will be seen that the maximum of negative velocity falls practically in the middle of the period, being at phase 63 days (about). At three points of the curve, namely at phases about 25, 51, and 106 days, there seem to be indications of changes of curvature, suggesting possibly a secondary wave superposed on the primary. Reference to this will be made later when the question of hydrogen absorption is considered.

It is interesting to draw attention to the fact that while the prismatic-camera form of spectroscope is not available for quantitative determinations of velocity in the line of sight, yet it is possible to obtain quite a good approximate value of the *period* of a binary when the hydrogen lines are of the double-bright nature. Further, if the observations are well distributed over the period, a good determination of the *form* of the velocity curve can also be ascertained.

Other Bright Lines.

In all good spectra of this star a large number of fine bright lines, which are all distinctly double, can be clearly seen ; they are evidently very similar in structure to the hydrogen lines, but are not superposed on broad absorption bands (see Plate 17). That they behave in a similar way to the hydrogen lines is without question, because when

* *Astr. Nach.*, 4442.
† *Lick Obs. Bull.*, **7**, 177, 1913.
‡ *Pub. All. Obs.*, **3**, 31, 1913.

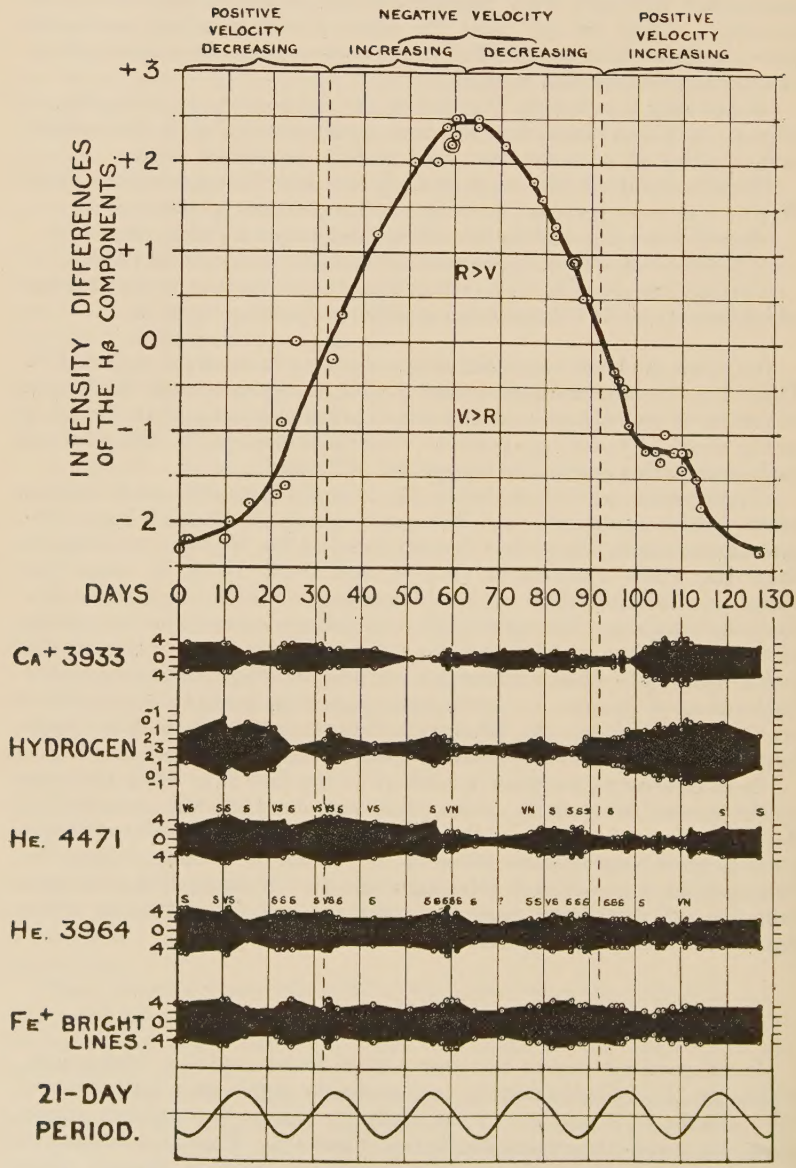


Fig. 2.—Variation of spectral line intensities throughout a whole period.

the violet component is brighter than the red component in any of the hydrogen lines so the violet components are brighter than the red components in those other bright lines, and *vice versa*.

In order to determine the wave-lengths and origins of these lines, a comparison was made with the spectrum of α Cygni taken with the same instrument. The lines here dealt with are only those the presence of which is undoubted, because while there were indications of other lines they might be subject to suspicion.

Table II. gives a list of the wave-lengths of these bright lines, and their intensities are added under the three headings: very strong (VS), strong (S), and weak (W). The corresponding wave-lengths, intensities, and origins of the α Cygni lines are given in the last four columns, and were extracted from a communication to the Royal Society by Sir Norman Lockyer and Mr. F. E. Baxandall.*

The striking result of this inquiry is that of the twenty-five lines dealt with, eighteen of them are coincident with the strongest ionised lines of iron. Of the other lines, λ4308·1, λ4330·7, λ4417·88, and λ4805·25 are probably due to ionised titanium, but are not the strongest ionised titanium lines: a line at λ4667·2 has no certain origin: a line at λ4731·7 is an unknown line and only recorded in α Cygni according to Baxandall.†

The ionised calcium line at λ3933·8 appears occasionally as a double bright line, or, as it might be otherwise described, an absorption line with bright borders. No reference can be found to show that this line has been described like this before, although several references mention only the absorption line of calcium at this wave-length, as will be mentioned later.

Plate 16 shows a spectrum of φ Persei with α Cygni for comparison, and some of the stronger bright ionised lines are indicated in the space between them.

Miss Maury seems to have been the first to have noticed the double character of the bright lines in this star, apart from the hydrogen lines. In her note on the spectrum ‡ she states that "the star further agrees with γ Cassiopeiæ in the bright bands and spaces seen elsewhere in the spectrum, some of these being even more distinct than in γ." In the spectrum of γ Cassiopeiæ Miss Maury seems to differentiate between two different types of bright lines. One kind she refers to as "bright bands," and the other kind as "not well defined bright bands . . . but have rather the nature of bright spaces between dark lines or of bright borders to these lines." The first type, according to her, are seen on the red side of H_β and the other on the violet side between H_β and H_γ. There is little doubt that the only difference between these two kinds of lines is that those on the red side of H_β are not separated so distinctly into two components as those on the violet side. In the particular regions where she specifies the appearance of the most prominent lines of each type, there are corresponding strong ionised iron lines.

* *Phil. Trans. R.S.*, **201**, 212, 1903.
† *Monthly Notices*, **83**, 171.
‡ *Harv. Ann.*, **28**, 101, Part 1, 1897.

TABLE II.

(*Bright Lines.*)

φ Persei. Lockyer. λ. (1).	φ Persei. Lockyer. I. (2).	φ Persei. Jordan. λ. (3).	γ Cass. Curtiss. λ. (4).	γ Cass. Curtiss. I. (5).	o Ceti (Compn.) Joy. λ. (6).	α Cygni. Lockyer and Baxandall. λ. (7).	I. (8).	Origins. (9).	
3933·8	S				3933·8	3933·8	10	3933·8	Ca+
4173·5	S					4173·5	6–7	4173·52	Fe+
								4173·70	Ti+
4178·9	S	4177·1	4177·4	3		4179·0	6–7	4178·95	Fe+
4233·3	VS	4233·2	4233·6	5	4233·2	4233·8	8	4233·33	Fe+
4308	W		4307·8	4		4308·1	4	4308·10	Ti+
4330·7	W					4330·7	2	4330·50	Ti+
								4330·87	
4351·9	S	4353·0	4353·3	4	4351·8	4351·9	7	4351·93	Cr. Fe+
								4352·08	Mg
4385·5	W		4384·3	5		4385·5	5–6	4385·55	Fe+
4417·9	W		4418·8	4	4416·8	4417·9	2–3	4417·88	Ti+
4489·0	S					4489·0	3	4489·35	Fe+
4491·6	S	4490·8	4491·5	2		4491·6	3–4	4491·57	Fe+
4508·5	W				4508·3	4508·5	5	4508·46	Fe+
4515·5	W				4515·3	4515·5	5	4515·51	Fe+
4520·4	W	4519·5			4520·2	4520·4	4	4520·40	Fe+
4522·7	W		4521·8	3	4522·6	4522·7	5	4522·69	Fe+
4549·8	S	4549·2			4549·5	4549·8	7	4549·64	Fe+
					4549·6			4549·81	Ti+
4556·1	S	4555·3	4557·5	4	4555·9	4556·1	5	4556·06	Fe+
								4556·31	Fe
4576·5	W					4576·5	3–4	4576·51	Fe+
4584·0	VS	4583·7	4583·8	5·7	4583·8	4584·0	7	4584·02	Fe+
4629·6	VS	4629·4	4629·3	6		4629·6	5–6	4629·52	Co. Ti. Fe+
4667·2	W	4666·7	4668·4	6		4667·2	2–3	4667·63	? Fe
								4667·77	? Ti
4731·7	W					4731·7	3–4		?
4805·2	W					4805·2	2	4805·25	Ti+
4924·1	VS		4925·7	5	4924·0	4924·1	5	4924·11	Fe+
5018·6	VS		5018·3	4	5018·5	5018·6	4	5018·63	Fe+

Column (2).—S=Strong. VS=Very strong. W=Weak.

Further, in the spectrum of γ Cassiopeiæ the hydrogen line at H_β is barely separated into two components while H_γ consists of two distinct components: it seems to be the rule for the amount of separa-

tion of the bright ionised lines to conform to that of the hydrogen lines in their respective parts of the spectrum.

In the spectrum of φ Persei the components of H_β are well separated, and this applies also to the ionised iron lines on the red side of H_β and elsewhere.

Miss Maury gives no actual list of the wave-lengths of these bright lines.

Merrill * has published a list of the chromospheric lines in φ Persei. He examined the spectrum photographically in the so-called visible region with a one-prism spectrograph attached to the 36-inch refractor. He only records one bright line between H_β and H_γ, namely, λ4629·52, a very strong ionised iron line. On the red side of H_β (within the region of Table II.) he records the bright lines at λλ4924·11 and 5018·5, both strong ionised iron lines. He noted that many of the lines were apparently double.

Jordan † also has published a list of wave-lengths of the emission lines, and writes that their character "when they are definite enough to show any structure, seems to be the same as that of the hydrogen lines: a broad emission with central absorption."

Table II., column (3), gives the wave-lengths he measured, but it should be noted that his spectrum extends towards the violet, commencing only with H_β (λ4861·5). It will be seen that all his measures approximate closely to the ionised iron lines. The only mention Jordan makes about the ionised calcium line at K is that "the K line is measurable on nearly all the plates, but gives very inconsistent velocity results."

It is a curious fact that Ludendorff ‡ makes no mention of bright lines other than those of hydrogen. He draws attention to the appearance of many faint absorption lines in the spectrum which in his estimation vary in intensity, so that at one time a certain line is present and soon after absent. Several of these absorption lines, according to him, frequently appear double (See later section under heading "Absorption Lines.")

Changes in the Intensities of the Bright Ionised Lines.

The Ionised Iron Lines.—The fact that the double bright ionised iron lines seem to appear stronger in some photographs than in others led Jordan § to examine his plates to find out whether any periodicity was connected with them. His result can best be briefly stated by the following extract from his paper:—

"On some plates they do not appear, but their visibility depends not upon the phase but upon the character of the exposure. Weak or very dense spectra do not in general show them. They are in the majority of cases very faint and indefinite, more like bright bands than lines."

* *Lick Obs. Bull.*, No. 224, p. 98, 1912.
† *Pub. Ast. Obs.*, **3**, 35, No. 5, 1913.
‡ *Astr. Nach.*, 4442.
§ *Pub. Ast. Obs.*, **3**, 34, 1913.

The above description of the lines cannot be said to apply to that of the lines in the Sidmouth photographs. In these the double lines are quite sharp, definite, and well separated, and could not in any instance be described as "bands." They appear in all the photographs that have been taken and, of course, are more prominent in the better photographs. That they appear fairly strong at all phases can be gathered from Plate 17, where photographs at six prominent phases are reproduced.

A critical examination of all the Sidmouth negatives was undertaken to detect any possible periodicity. The inquiry was not easy because all the other lines in the spectrum of this star are so variable throughout the various phases that there is no basis on which their intensities could be founded. The method that was finally adopted was simply to judge their general conspicuousness in the spectrum as a whole.

Five categories of intensity were chosen, namely (5) very strong, (4) strong, (3) moderate, (2) weak, and (1) very weak, and in all cases the character of the lines was sharp.

The result of this investigation is tabulated in the last column of Table V. under the heading Fe^{+}, column (9), arranged in order of phase, and those values are shown graphically near the bottom of fig. 2.

It will be seen that while the lines are always fairly strong, there are indications of changes in intensity which may or may not be real, depending on whether they can be associated with those of other lines. Reference to this will be made later.

The Ionised Calcium Line.—In the best negatives the ionised calcium line at $\lambda 3933{\cdot}8$ is decidedly an absorption line with bright borders, but as a rule the absorption line is much more conspicuous than the bright borders. In studying any possible change with phase it has been regarded as an absorption line only. Its intensity has been determined similarly to that of the ionised iron lines, the same grades being adopted. This line is practically always sharp.

That it is subject to a distinct variation in relation to phase is clearly seen from the figures given in Table V., column (3), and graphically shown in fig. 2 just under the curve.

The line is continuously weak or very weak during the whole period of negative velocity and moderate to strong during the positive velocity phase, seeming to rise rapidly from weak to very strong during the interval of 18 days from the commencement of the positive velocity phase. Other minor fluctuations are present.

The ϕ Persei Bright Lines in other Stars.

In photographing the spectra of bright-hydrogen-line stars at Sidmouth it has been noticed that many of these stars exhibit bright lines similar in position to those recorded in ϕ Persei.

Thus γ Cassiopeiæ shows them nearly equally as strong and also double, but not so widely separated or so sharp as in ϕ Persei. They appear also double and nearly as bright in ψ Persei and weaker in ζ Tauri. They are weaker also in π Aquarii and 25 Orionis. In H.D. 20336 they are just visible.

Thus in all these stars the strongest bright lines are those of ionised iron.

That the bright lines, as exhibited in γ Cassiopeiæ, are due to ionised iron has been pointed out by Baxandall.*

While he published no list of wave-lengths, he pointed out that "There is distinct evidence that the chief bright lines, other than those of hydrogen, are identical in position with some of the stronger lines in α Cygni, which have been previously identified with the enhanced lines of various metals. Of these particular metals iron seems to be better represented in the bright lines of γ Cassiopeiæ than the others."

It is interesting to note that while Baxandall states that for the above investigation "the most satisfactory negatives show more lines than were recorded in the (previously) published paper," no mention is made of the duplicity of the lines, but only the statement "the lines in γ Cassiopeiæ are nearly all ill-defined and hazy." It may be mentioned here that the bright components of the hydrogen lines in φ Persei are well separated and so are the bright components of the ionised lines. In the case of γ Cassiopeiæ the Sidmouth photographs show the bright components of H_β scarcely separated and those of H_γ just separated and the bright lines between H_β and H_γ as very close doubles, and not "ill-defined and hazy."

While many other observers have investigated the spectrum of γ Cassiopeiæ, brief reference will only be made to the detailed investigation published by Ralph H. Curtiss † in 1915. Curtiss noticed that the structure of the hydrogen lines extended also to the emission lines. His measured wave-lengths corresponding to those in Table II., column (1), are added in column (4), and it will be seen that he also identifies most of the bright lines with those of ionised iron.

Another bright-hydrogen-line star exhibiting these lines prominently is μ Centauri. Thus Sir Norman Lockyer and Mr. Baxandall ‡ stated in 1905:—"A close examination has now shown that nearly all the most marked bright lines in μ Centauri—other than those of hydrogen—occupy positions closely corresponding to those of the most conspicuous enhanced lines of iron."

Quite recently (1924) Merrill § has referred to the star H.D. 161114 as an "iron star." This star has also bright hydrogen lines. From observations made during the last few years at Mount Wilson, he states that seventy bright lines in the blue and violet have been revealed and about twenty-five of the lines, including most of the stronger ones, are due to iron. He further remarks:—"A remarkable fact is that the *enhanced-line* spectrum is present in greater purity than in any laboratory source."

No less interesting is the still more recently recorded fact by Joy ‖ (1924) that the spectrum of the close companion of ο Ceti, in addition to the bright hydrogen lines, displays other bright lines nearly all of

* *Pub. Solar Physics Committee*, 1914, "The Spectrum of γ Cassiopeiæ."
† *Pub. Astr. Obs. Michigan*, **2**, 11 *et seq.*
‡ *Roy. Soc. Proc.*, **74**, 548, 1905.
§ *Pub. Astr. Soc. Pac.*, **36**, 225, 1924.
‖ *Ibid.*, **36**, 290, 1924.

which comprise the stronger lines of ionised iron in that part of the spectrum which has been photographed.

The wave-lengths he has published are given in Table II., column (6). It is curious to note that quite a strong ionised iron line at λ4629·6 is missing from his list.

Joy further records the fact that "the H and K lines of calcium are also bright and rather wide in the companion (of o Ceti), but have not been observed in H.D. 161114."

Cannon,* in his work on the orbit of φ Persei, used the K line in his measurements. He does not make any reference to bright lines in relation to it, but describes the absorption line as "seeming to appear when the star was in the ascending slope of the velocity curve" and "very faint." This latter statement is verified in the present investigation, Cannon's "ascending slope" corresponding to phases 80 to 130 days in fig. 2.

Thus we have the following list of stars which exhibit the predominance of ionised iron in their bright lines other than those of hydrogen. The spectral types are taken from the Henry Draper Catalogue, with the exception of H.D. 161114 and the companion to o Ceti, which have been stated tentatively by Joy.†

The stars are placed below in order of spectrum type; they are not necessarily in the order of intensity of the bright lines.

γ Cas.	B0p	ζ Tau.	B3p
φ Per.	B0p	25 Ori.	B3p
π Aqr.	B1p	ψ Per.	B5p
μ Cen.	B2p	H.D. 161114	B8p
H.D. 20336	B3p	o Ceti (Comp.)	B8p.

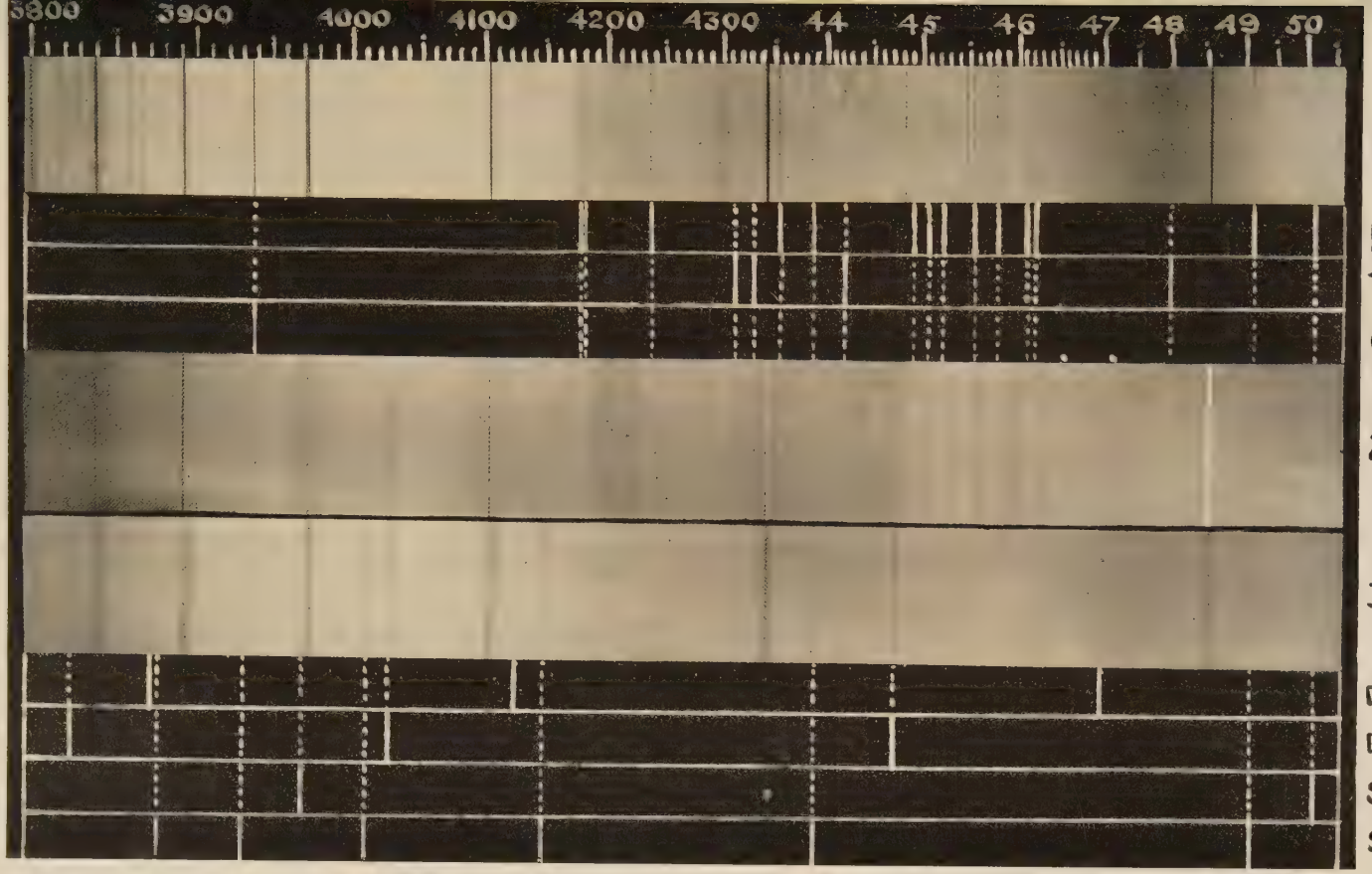
The Absorption Lines.

In addition to the bright hydrogen and ionised metallic lines in the spectrum of φ Persei only a few absorption lines can be seen. Some of these appear sometimes as very sharp lines and sometimes as very nebulous lines at different velocity phases. All these lines appear strong and sharp in the spectrum of γ Orionis, a typical helium spectrum star.

Table III. gives a list of the wave-lengths of these φ Persei lines, together with their intensities (S=strong, W=weak), in columns (3) and (4). They are all, with one exception, due to helium, and column (8) shows the particular series to which, according to Fowler,‡ they belong. The exceptional line referred to above is the ionised magnesium line at λ4481·3, which occasionally appears as a very weak line.

Plate 16 shows a spectrum of φ Persei with γ Orionis as comparison, and below them are marked the different series of helium lines observed, all of which are nebulous in this particular reproduction of the star. The letters on the right-hand side denote: D.S., Doublet, Sharp; D.D. Doublet, Diffuse; S.P., Singlet, Principal; and S.D., Singlet, Diffuse.

* *Journ. Astr. Soc. Canada*, **4**, 195, 1910.
† *Pub. Astr. Soc. Pac.*, **36**, 291, 1924.
‡ *Report on Series in Line Spectra*, pp. 93, 94, 1922.



1. α Cygni. 2. φ Persei. 3. γ Orionis.

Previous observers of this star, excluding Miss Maury, have recorded a much less number of absorption lines than appear in the Sidmouth photographs.

Thus Cannon * only refers to the line λ4472, which he describes as "faint."

Merrill † gives in his table the three helium lines at λλ4471·8, 5015·7, and 5876·0, together with the sodium double at λλ5890·18 and 5896·16. All these, with the exception of the first line, are outside the photographic region of the Sidmouth plates.

Jordan ‡ only refers to three lines. He finds the helium line at λ3965 visible on the majority of his plates, but those at λλ4026 and 4472 visible and measurable on "only a few plates at certain phases, especially from twenty to forty days after maximum positive velocity." The Sidmouth photographs corroborate this statement, as will be shown further on.

Miss Maury § states that "the dark bands or lines present are nearly the same as in γ Cassiopeiæ," and she gives a list of some of the Orion lines which are absent and others which are "faint or invisible."

In Table III., column (5), only those lines recorded by her which appear in the Sidmouth photographs have been inserted. There are eight other absorption lines observed by Miss Maury and not shown here. It is strange that the two strong helium lines at λλ3964·7 and 4921·9 are omitted from her list, especially in the case of the former, which is never absent from the spectrum at any phase.

Miss Maury makes no mention of the ionised magnesium line at λ4481·3.

In a previous paragraph it has been pointed out that, in the bright-hydrogen-line star ζ Tauri, the bright ionised metallic lines are similar to, but fainter than those in φ Persei. In an earlier paper ‖ a list of the absorption lines in this star was published, and this showed that (p. 331) "all the broad and diffuse lines are due to helium, in which the double-diffuse, and double-sharp and also the single-principal and single-diffuse series are represented."

It will be seen, therefore, that these two stars indicate the presence of helium in a similar way, that is by the presence of the same series. The ζ Tauri wave-lengths are inserted in Table III., column (6), for comparison, while those for γ Orionis, taken from the South Kensington observations,¶ are given in the first two columns.

It will be noticed that the wave-lengths are not given in the usual order, but grouped according to their appearance in the different series of helium. The ionised line of magnesium, the only other absorption line found, is placed at the bottom of the table.

* *Journ. Roy. Astr. Soc. Canada*, **4**, 195, 1910.
† *Lick Obs. Bull.*, **7**, 98, 1912.
‡ *Pub. All. Obs.*, **3**, 34, 1913.
§ *Harv. Ann.*, **28**, 101, Part 1, 1897.
‖ *Monthly Notices*, **83**, 329.
¶ *Solar Physics Observatory Publication*, "Catalogue of 470 Brighter Stars," p. 44 *et seq.*, 1902.

TABLE III.

(*Absorption Lines in φ Persei.*)

γ Orionis. Kensington.		φ Persei.			ζ Tauri.	Origin.	Helium Series.		
		Lockyer.		Maury.	Lockyer.				
λ.	I.	λ.	I.	λ.	λ.				
(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).		
4713·3	4	4713·1	W	4712·8	4713·3	He	Doublet System, Sharp.		
4121·0	5	4120·8	W	4120·5	4120	,,	,,	,,	,,
3868·0	4	3867·5	W	3867·6	..	,,	,,	,,	,,
4471·6	8	4471·5	S	4471·8	4471·7	,,	,,	,,	Diffuse.
4026·3	8	4026·2	S	4026·4	4024·6 4028·5	,,	,,	,,	,,
3819·7	8	3819·6	S	3819·2	3819·7	,,	,,	,,	,,
3964·9	8	3964·7	S	..	3964	,,	Singlet	,,	Principal.
..	..	4921·9	S	..	4921·9	,,	,,	,,	Diffuse.
4388·1	6	4387·9	S	4387·8	4388	,,	,,	,,	,,
4143·9	7	4143·8	S	4144·0	4143	,,	,,	,,	,,
4009·4	6	4009·3	S	4009·5	4009	,,	,,	,,	,,
3926·7	4	3926·5	W	3927·1	3926	,,	,,	,,	,,
3871·6	4	3871·8	W	..	3872·4	,,	,,	,,	,,
4481·3	4	4481·3	W	..	4481·3	Mg^+			

Column (4).—S=Strong. W=Weak.

It cannot be stated whether the line at λ3888·7, belonging to the Principal series of the helium doublet system, is present, as it is so close to the hydrogen line H_ϵ (λ3889·0). While four representations of the sharp series of the singlet system are recorded in the spectrum of γ Orionis, none of these can be seen in the spectrum of φ Persei.

Changes in the Intensities of the Absorption Lines.

As the bright components of the hydrogen lines and also those of the bright ionised metallic lines change their intensities with change of phase, so it is quite possible that the absorption lines may vary their intensities with change of phase also.

That the variability of the absorption lines has been suspected and recorded is shown by the remarks on the spectrum of this star which appear in the Henry Draper Catalogue.*

It is stated there that "the helium lines are strong at the intermediate phase, only suspected to be present at time of maximum positive velocity, and are not visible at maximum negative velocity.

* Vol. **91**, p. 280, 1918.

At the latter phase the dark hydrogen lines are wide and faint. At the intermediate and positive velocity phases the dark hydrogen lines are very narrow and sharply defined."

The Sidmouth photographs show that both the hydrogen and helium absorption lines also are by no means constant in intensity throughout a period but exhibit distinct changes according to phase.

The behaviour of these gas lines and also of the ionised magnesium line are treated separately below under respective headings.

The Absorption Lines of Helium.—The two strongest lines of the diffuse doublet system of helium, namely λλ4471·5 and 4026·3, were found to behave in a similar manner, so the following remarks apply to them both.

Their intensities were determined after the same manner as the ionised calcium line at λ3933, and their character was noted as to whether it was sharp (S) or nebulous (N): the letters VS represented instances when the lines appeared sharper than normal. (These signs are used in fig. 2.)

Table V., columns (5) and (6), summarises the results in order of phase. It will be seen that there is a considerable variation of a periodic nature. This variation is rendered more apparent in the graphical representation shown below the velocity curve in fig. 2.

At the maximum positive velocity phase the lines are moderately strong and inclined to be somewhat nebulous.

They then increase in intensity and sharpness, and as they reach the following intermediate phase they become exceedingly strong and sharp and are the most prominent absorption lines in the whole spectrum excepting those of hydrogen. This agrees with the remarks of Jordan, who, as previously stated, noticed that these lines were visible and measurable at certain phases, "especially from twenty to forty days after maximum positive velocity."

Approaching the maximum negative velocity they become nebulous again, but are still strong. About the time of maximum negative velocity they become, for a short time, very strong, and then suddenly weaken, increasing their intensity slightly up to about ten days before the next intermediate phase. They then become moderate and weak again until they take on the appearance as described above at the next phase of maximum positive velocity.

Thus the behaviour of these lines does not quite conform to the description as stated in the Henry Draper Catalogue.

Again the other helium line that appears in the Sidmouth spectra is that in the Principal Singlet System at λ3964·7. The other line of that series is at λ5015·7, but unfortunately it is not in a good photographic portion of the spectrum on the plates used, and so cannot be discussed.

This line at λ3964·7 also by no means maintains the same intensity throughout all phases, although, unlike λ4471, it is always moderately strong, as was stated by Jordan.

The intensities and characteristics in the various negatives are given in Table V., columns (7) and (8), in order of phase. They are also graphically represented in fig. 2 under the velocity curve.

The line is moderate to strong from maximum positive to maximum negative velocity, showing a slightly weakening feature before the latter phase. It then suddenly weakens, but gradually strengthens and reaches a maximum about six days before the next intermediate phase is reached, after which it diminishes on the average in intensity.

The Absorption Lines of Hydrogen.—A most noteworthy feature that can be observed by examining different spectra of ϕ Persei as a whole is that, while H_ζ does not seem to change in intensity at all in any of the negatives covering all phases, some spectra exhibit the double bright line H_β and the central absorption of H_ζ as the most prominent and outstanding lines, while others display the central absorptions of H_γ, H_δ, H_ϵ, H_η, and H_θ so conspicuously that H_ζ is no longer specially noticeable.

This is due to the fact that the narrow absorption lines of hydrogen, superimposed on the bright bands, appear more conspicuous by becoming broader at some times than at others.

It was considered, therefore, of great importance to try, if possible, to measure in each negative the intensity of the central absorption line of H_ζ against those of H_δ and H_ϵ to detect whether any cyclical change was taking place in the amount of hydrogen absorption.

To obviate any bias on the writer's part the negatives were given to Mr. Edwards in a miscellaneous order to determine the differences of intensities by means of the wedge method with the micrometer.

To determine the difference of intensity between two lines in a spectrum by this method it is usual to measure only lines fairly close together in order not to bring in any error which might arise from different intensity of the continuous spectrum about the lines. In this case, however, the differences to be measured were so large that any small change in the intensity of the continuous spectrum about each line could be practically neglected.

All the measures were reduced to one cycle and arranged in order of phase as shown in the accompanying table (Table IV.). The measures are given in detail as they not only show how the striking similarities of the two sets of differences were obtained, but they bring out the great use of the wedge method in detecting such differences.

TABLE IV.

(*Variation of Hydrogen Absorption.*)

Phase.	Neg. No.	Intensities.			Differences.		Differences from Means.	
		H_ζ.	H_ϵ.	H_δ.	$H_\zeta - H_\epsilon$.	$H_\zeta - H_\delta$.	$H_\zeta - H_\epsilon$.	$H_\zeta - H_\delta$.
(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).	(9).
1	611	..	..	..	+1·8	+0·7	+0·5	+0·4
2	536	18·3	17·0	18·1	+1·3	+0·2	0·0	−0·1
10	472	19·1	19·3	20·2	−0·2	−1·1	−1·5	−1·4
11	537	18·8	17·8	18·8	+1·0	0·0	−0·3	−0·3
15	475	17·9	17·8	18·6	+0·1	−0·7	−1·2	−1·0
22	483	15·8	15·1	16·0	+0·7	−0·2	−0·6	−0·5
23	544	17·2	16·4	17·1	+0·8	+0·1	−0·5	−0·2

TABLE IV.—*continued.*

Phase.	Neg. No.	Intensities.			Differences.		Differences from Means.	
		H_ζ.	H_ϵ.	H_δ.	$H_\zeta - H_\epsilon$.	$H_\zeta - H_\delta$.	$H_\zeta - H_\epsilon$.	$H_\zeta - H_\delta$.
(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).	(9).
25	**403**	**18·8**	**15·8**	**16·8**	**+3·0**	**+2·0**	**+1·7**	**+1·7**
32	**548**	**18·8**	**16·8**	**17·6**	**+2·0**	**+1·2**	**+0·7**	**+0·9**
33	**550**	**17·5**	**16·5**	**17·4**	**+1·0**	**+0·1**	**−0·3**	**−0·2**
35	489	17·2	15·6	16·6	+1·6	+0·6	+0·3	+0·3
43	555	17·0	14·5	15·7	+2·5	+1·3	+1·2	+1·0
51	558	19·0	16·8	17·6	+2·2	+1·4	+0·9	+1·1
56	564	18·3	17·0	18·1	+1·3	+0·2	0·0	−0·1
58	355	18·3	16·7	17·8	+1·6	+0·5	+0·3	+0·2
59	362	18·2	16·2	17·3	+2·0	+0·9	+0·7	+0·6
60	570	17·5	15·0	16·3	+2·5	+1·2	+1·2	+0·9
60	571	18·0	15·5	16·4	+2·5	+1·6	+1·2	+1·3
65	576	16·5	13·8	15·0	+2·7	+1·5	+1·4	+1·2
71	577	17·8	15·3	16·4	+2·5	+1·4	+1·2	+1·1
77	578	17·0	15·0	16·0	+2·0	+1·0	+0·7	+0·7
79	581	16·8	15·0	16·1	+1·8	+0·7	+0·5	+0·4
82	584	12·3	10·0	11·1	+2·3	+1·2	+1·0	+0·9
86	585	18·0	15·5	16·5	+2·5	+1·5	+1·2	+1·2
86	497	18·5	15·8	16·5	+2·7	+2·0	+1·4	+1·7
88	435	17·1	14·8	15·7	+2·3	+1·4	+1·0	+1·1
89	588	18·0	16·5	17·5	+1·5	+0·5	+0·2	+0·2
95	590	16·0	14·5	15·7	+1·5	+0·3	+0·2	0·0
96	592	17·2	15·7	16·8	+1·5	+0·4	+0·2	+0·1
97	500	18·0	16·7	17·7	+1·3	+0·3	0·0	0·0
98	595	16·2	15·6	16·4	+0·6	−0·2	−0·7	−0·5
102	597	16·7	16·0	17·2	+0·7	−0·5	−0·6	−0·8
104	598	16·8	16·4	17·6	+0·4	−0·8	−0·9	−1·1
105	601	17·7	17·7	18·1	0·0	−1·1	−1·3	−1·4
106	509	18·6	18·3	19·3	+0·3	−1·0	−1·0	−1·3
108	605	15·0	15·0	15·8	0·0	−0·8	−1·3	−1·1
110	510	18·7	18·2	19·0	+0·5	−0·3	−0·8	−0·6
110	606	16·8	16·8	17·7	0·0	−0·9	−1·3	−1·2
111	515	17·1	17·5	18·4	−0·4	−1·3	−1·7	−1·6
112	607	13·0	13·2	14·0	−0·2	−1·0	−1·5	−1·3
114	608	..	..	..	−0·2	−1·0	−1·5	−1·3
119	610	..	..	..	−0·2	−1·0	−1·5	−1·3
127	530	16·3	15·0	16·0	+1·3	+0·3	0·0	0·0
				Means	+1·3	+0·3.		

The first column shows the number of days elapsed from the zero phase, and column (2) the numbers of the negatives. Columns (3), (4), and (5) give the means of the measures of the intensities of the three hydrogen central absorption lines under the wedge, each mean being the result of a number of readings. Columns (6) and (7) indicate the differences between H_ζ and the respective lines H_ϵ and H_δ, and the last two columns show the differences from the respective mean values as placed at the bottom of columns (6) and (7).

Towards the latter end of March and the beginning of April of the present year the spectrum of ϕ Persei was difficult to photograph in consequence of its westward position and the interference of twilight. In three cases (phases 114^d, 119^d, and 1^d) the negatives were too thin for measurement in the micrometer. Direct comparisons of these were, however, made with other negatives which exhibited similar hydrogen absorption, and the values ascribed to the latter were adopted. Such negatives are given no readings of intensity in columns (3), (4), and (5) in Table IV., so they can be easily identified.

It will be seen at a glance that the differences in both the last columns form themselves into two main groups. The first extends from phase 25^d to phase 96^d, which includes all the positive values (with two exceptions), while the second group commences at phase 98 and terminates at phase 23^d, which includes all the negative values with only one exception.

The positive values denote that the absorption at H_ζ is much more intense than that of the other two lines.

On the other hand, the negative values show that the intensities of H_ϵ and H_δ indicate very strong absorption which nearly equalises and sometimes excels that at H_ζ.

Now the normal curve as illustrated in fig. 2 shows that the period of negative velocity extends from phase 32^d to 92^d. The period showing small hydrogen absorption, which extends from phase 25^d to phase 96^d, therefore coincides with, but is a little longer than, that of negative velocity.

The differences shown in the last two columns of Table IV. indicate quite clearly that a short-period variation, in addition to that of the long period, exists; but this is more apparent when they are displayed graphically. This has been done in fig. 3.

The upper curve shows the differences H_ζ minus H_ϵ, while the lower curve represents the differences H_ζ minus H_δ.

It will be seen from Table IV. that the latter differences are nearly always less than the former, but in fig. 3 the curve for the differences of H_ζ minus H_δ has been placed at a much lower level, and a special scale attached in order to further separate the curves.

It is interesting to note in the first instance the close similarity of the two curves, emphasising the fact that while the lines H_ζ, H_ϵ, and H_δ are large distances apart in the spectrum the wedge method of measurement has been capable of recording these similar differences.

In fig. 3 the horizontal broken lines indicate the mean values derived from columns (6) and (7) of Table IV. for each curve, and the relation of

the plus and minus signs in columns (8) and (9) to each other can be clearly seen.

While the individual measures, represented by small circles, are connected by short lines, an approximate smooth mean curve has been drawn to indicate the larger variation. There undoubtedly appears to be a variation of short period, with quite a considerable amplitude, superimposed on the primary variation. The only regular variation that seems to fit the measures is one having a period of 21 days or equal in length to a sixth of the primary. Such a short-period change is indicated in fig. 3 below the curves.

It was mentioned previously that the intensity or velocity curve illustrated in fig. 2 indicated three subsidiary changes of curvature

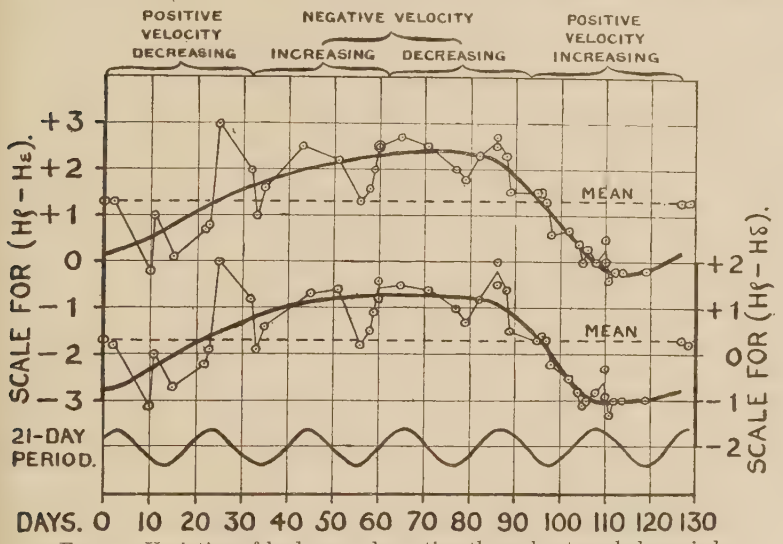


Fig. 3.—Variation of hydrogen absorption throughout a whole period.

superimposed on the primary curve. These took place in the neighbourhood of the phase days 25, 51, and 106: they may possibly correspond with the maxima of the 21-day period of the secondary occurring on the phase days 24, 45, and 108.

It is interesting to note that the researches of Ludendorff, Jordan, and Cannon, to which references have previously been made, have all recorded the subsidiary wave which appears about phase 106, but Cannon has considered this as only one of two subsidiary waves which appear during the complete cycle of the primary: the other wave, according to him, occurs some days before the maximum negative velocity, corresponding possibly to that at phase 25 shown in fig. 2.

The behaviour of the hydrogen absorption due to the primary corroborates to some extent the statement in the Henry Draper Catalogue previously mentioned. Where the hydrogen lines are there described as "wide and faint" they are here "very weak," while

where they were stated to be " very narrow and sharply defined " they are here " very strong."

In order to compare the behaviour of the hydrogen absorption lines with those of the other elements, the differences given in Table IV., column (6), have all been subtracted from 3, and the values thus obtained shown in Table V., column (4). These latter values are plotted below the velocity curve in fig. 2. The nature of the primary and secondary variation of the absorption can be clearly seen.

The Absorption Line of Ionised Magnesium.—The absorption line of ionised magnesium at $\lambda 4481{\cdot}3$ is practically on the verge of invisibility or absent in most photographs, and on only two photographs attains the intensity of 2 out of a maximum of 5.

There seems to be a tendency for the line to be more visible during the phase from 0 to 63, but after that until phase 0 again there are only three instances of visibility.

The Inter-behaviour of the Bright and Dark Lines during a Complete Cycle.

In previous paragraphs the variations of the emission and absorption lines have been stated individually in relation to the different phases throughout the whole period of 126·8 days.

It is proposed here to point out the inter-behaviour of these lines as deduced from all the measures reduced to one period.

The data regarding the intensities of the different lines have all been brought together in Table V. and graphically shown in fig. 2. This figure also shows the curve for the intensity differences of the components of H_β and the velocity phases placed at the top. At the bottom of the figure the 21-day period variation shown in fig. 3 has been here reproduced, but inverted to conform to the hydrogen absorption intensity displayed above it.

The horizontal bands of varying thickness in fig. 2 illustrate the changing intensities of the emission and absorption lines on scales previously described. The broader the bands the more intense are the lines. The thick vertical broken lines separate the two main phases of the star, namely, the periods of negative and positive velocity as deduced from the normal curve.

All the lines of Ca^+, H, and Fe^+ in the spectra were sharp, but the helium lines were sometimes sharp and at other times nebulous. In the figure in the case of the helium lines when the lines were sharp or very sharp they are indicated by the letters S or VS. All other points unmarked refer to nebulous lines, but the letters VN have been inserted when they were very nebulous.

The outstanding feature of these line intensities is that the absorption lines of hydrogen and calcium behave somewhat alike, both being very weak or exhibiting least absorption at the phase of negative velocity and greatest prominence at the phase of positive velocity.

The diffuse doublet system of helium, as indicated by the line at $\lambda 4471$, seems to behave rather differently, being most strong and sharp in the phase between maximum positive and maximum negative

velocity, and most weak and nebulous between maximum negative and maximum positive velocity. The helium line at λ3964 of the principal singlet system behaves somewhat similarly as regards strength and nebulosity, but never becomes so weak as the lines of the diffuse series.

TABLE V.

(*Intensities of Bright and Dark Lines.*)

Phase.	Neg. No.	Ca^+ 3933.	Hydrogen.	He 4471.		He 3964.		Fe^+.
		I.	I.	I.	Ch.	I.	Ch.	I.
(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).	(9).
1	611	2	1·2	3	N	5	S	4
2	536	3	1·7	3	VS	5	S	4
10	472	3	3·2	5	S	4	S	5
11	537	3	2·0	5	S	5	VS	5
15	475	1	2·9	4	S	2	N	3
22	483	2	2·3			4	S	3
23	544	3	2·2	4	VS	4	S	4
25	403	3	0·0	3	S	4	S	5
32	548	3	1·0	5	VS	4	S	3
33	550	2	2·0	5	VS	4	VS	5
35	489	2	1·4	5	S	3	S	3
43	555	2	0·5	4	VS	3	S	4
51	558	0	0·8	3	N	3	N	3
56	564	0	1·7	4	S	4	S	4
58	355	1	1·4	1	N	4	S	4
59	362	2	1·0	2	N	5	S	5
60	570	1	0·5	2	VN	3	S	5
61	571	1	0·5	2	N	4	S	5
65	576	1	0·3	1	N	2	S	3
71	577	2	0·5	1	N	2		3
77	578	2	1·0	2	VN	3	S	4
79	581	1	1·2	3	N	3	S	4
82	584	2	0·7	3	S	4	VS	5
86	497	1	0·3	2	S	3	S	5
86	585	2	0·5	3	S	4	S	5
88	435	1	0·7	3	S	3	S	4
89	588	1	1·5	2	S	4	S	4
95	590	1	1·5	1	S	3	S	4
96	592	0	1·5	1		3	S	4
97	500	2	1·7	1	N	3	S	4
98	595	0	2·4	2	N	3	S	3
102	597	3	2·3	1		2	S	3
104	598	4	2·6	1	N	2	N	2
105	601	4	3·0	2	N	3	N	3
106	509	5	2·7	1		3	N	3
108	605	4	3·0	1	N	2	N	4
110	510	5	2·5	1	N	2	N	3
110	606	4	3·0	1	N	2	N	..
111	515	5	3·4	1	N	3	VN	4
112	607	4	3·2	3	S	2	N	..
114	608	..	3·2	2	S	3	S	4
119	610	..	3·2	3	S	3	S	..
127	530	3	1·7	3	S	3	S	4

Columns (6) (8).—S=Sharp. N=Nebulous. V=Very.

With regard to the bright ionised iron lines there does not seem to be any very definite variation of intensity throughout a cycle. Their behaviour is, however, distinctly unlike those of hydrogen and Ca^+, but seems to approximate somewhat to that of the helium line λ3964.

All the bands seem to suggest that there is an underlying short-period variation at work, as is most prominently indicated by the hydrogen band, but the indication is not very distinct owing probably to the insufficient number of photographs taken at very short intervals of time.

It is hoped that, when the star again comes into a convenient position for observation, a much more complete series of photographs will be secured, now that it is known that such rapid changes are taking place in its spectrum.

φ Persei as a Variable Star.

It has been pointed out on a preceding page that the central absorption of the hydrogen lines varies to a very great extent during a cycle of φ Persei, being most prominent at the phase embodying positive velocity and most weak at that of negative velocity.

This absorption strengthens very rapidly in the hydrogen lines from H_γ towards the ultra violet, and it is so strong that when the absorption is at a maximum for the whole spectrum the star would probably be photographically fainter than when the absorption is at a minimum.

In fact the star should be a variable with a period of 126·8 days, having an amplitude of variation of probably from two- to three-tenths of a magnitude. Its maximum brightness should occur at the epochs of maximum negative velocity.

A search has been made to find out whether this star has been previously stated to be undergoing small variations of brightness, but without success.

The star has been considered constant in magnitude, and its magnitude is given as 4·19 in the Revised Harvard Photometry.*

The above volume also publishes a series of residuals between the Harvard determination and those of other catalogues. These residuals, converted into magnitudes, are given below and compared with the Harvard mean :—

Sir William Herschel	4·40
Uranometria Oxoniensis	4·29
Potsdam Catalogue	4·27
Bailey's Uranometry	4·18
Harvard Revised	4·19

From the above it will be seen that there are quite considerable differences of magnitudes indicated.

An attempt was made to examine the original Harvard photometric observations to find out whether any variation of brightness could be detected from the visual observations, in spite of the fact that any variation would be more pronounced in photographic observations.

In the Harvard volume the mean magnitude of the star is given as 4·19, this being a mean of four means. Each of the last mean values is a mean of several observations.

In the following table (Table VI.) the first three columns illustrate how the final mean value of 4·19 is obtained.

Now it will be seen that not only is the third value in the second column very different from the other three values in the same column, but that the values in the third column differ very greatly among themselves. Thus, taking the extreme values 3·80 and 4·41, there is here a difference of six-tenths of a magnitude, which seems very excessive if the star be of constant magnitude.

It was considered probable that if φ Persei were a variable star of small amplitude such differences might be explained by the individual

* *Ann. Astr. Obs. Harvard Coll.*, **50**, 27, 1908.

observations being made at different phases. An attempt was therefore made to reduce all the original observations into one velocity-period of the star, basing zero phase on Jordan's date of maximum positive velocity, namely 1909 December 29 (J.D. 2418670), and the period of 126·8 days derived in an earlier part of this paper.

Table VI. thus shows in columns (4) and (5) the Julian day of each Harvard observation followed by the Julian date of the epoch of maximum positive velocity immediately preceding that observation. Column (6) gives the deduced phases in days from the epoch of maximum positive velocity.

It has been shown in a former paragraph that the hydrogen absorption is greatest at the phase of positive velocity and least at negative velocity, so that it would be expected that ϕ Persei should be brighter at the latter phase than at the former. The phase of negative velocity covers the period from phase 32 days to phase 92 days: it therefore would be expected that the mean of all the values of magnitude made during this interval would be smaller than the mean of those made at the other phase.

The last two columns of the table show the magnitudes separated into two columns, according as they belong to the phase of negative (column (7)) or positive (column (8)) velocity, and it will be seen that their means are identical.

TABLE VI.

(*Magnitudes of φ Persei.*)

Mean Mag.	Means. Mag.	Indl. Obs. Mag.	Julian Date of Observation.	Julian Date of Epoch preceding Max. Pos. Vel.	Phase Days.	Velocity. Neg.	Velocity. Pos.
(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).
		4·1	240 7710	240 7638·4	71·6	4·1	
		4·0	8022	8018·8	3·2		4·0
		4·2	8043	,,	24·2		4·2
	4·24	4·2	8381	8272·4	108·6		4·2
		4·2	8389	,,	116·6		4·2
		4·3	8391	,,	118·6		4·3
		4·27	240 9473	240 9413·6	59·4	4·27	
	4·27	4·27	9487	,,	73·4	4·27	
		3·80	241 2078	241 2076·4	1·6		3·8
41·9		4·10	2449	2330·0	119·0		4·1
		4·10	2775	2710·4	64·6	4·1	
	4·05	4·20	2778	,,	67·6	4·2	
		4·20	2786	,,	75·6	4·2	
		4·00	2790	,,	79·6	4·0	
		4·00	241 3131	241 3090·8	40·2	4·0	
	4·20	4·20	3169	,,	78·2	4·2	
		4·41	4223	4104·1	118·9		4·41
					Means	4·15	4·15

The reduction of these observations does not therefore indicate the variability that was expected.

Since more accurate methods are now available to determine photographically small changes of magnitude, arrangements have been made, outside of the Norman Lockyer Observatory, to observe this star with up-to-date photometric apparatus throughout the next convenient complete period.

Summary.

1. This paper contains a discussion of 55 plates, 51 of which have been taken since 1924 August 8, of the spectrum of the bright-hydrogen-line star φ Persei (type Bope).

2. Measures of the relative intensities of the bright components of the hydrogen lines H_β and H_γ result in the determination of the length of the period of variation, namely 126·8 days.

3. Previous determinations, by measures of the velocity in the line of sight by slit spectroscopes, gave periods of 126·5 and 126·6 days.

4. The curve showing the variations of the relative intensities of the bright components of the hydrogen lines corresponds to the velocity curve, the points of maxima of the former coinciding with the epochs of maximum negative velocity of the latter.

5. Numerous other bright lines in the spectrum are all double and the relative intensities of their components behave like the hydrogen lines. They are mainly due to ionised iron.

6. A list is given of ten other B-type stars which exhibit the peculiarity of having ionised iron as the origin of their brightest metallic emission lines.

7. It is shown that all the absorption lines in the spectrum are due to lines in the singlet and doublet systems of helium with the exception of one, the origin of which is ionised magnesium (λ4481·3).

8. The helium lines become alternately sharp and nebulous at different phases of the velocity.

9. Changes in the intensities of the emission and absorption lines are discussed and are found to vary with the change of phase of the velocity in the line of sight. Of special interest is the variation of the hydrogen absorption lines.

10. It is suggested that, underlying the period of 126·8 days, there is a secondary one of 21 days, or about one-sixth of the primary.

11. It is suggested that φ Persei is a variable having an amplitude of variation of two- to three-tenths of a magnitude, but a discussion of the published observations of magnitude does not indicate any *periodic* variation. Differences between extreme values of observed magnitude amount, however, to as much as six-tenths of a magnitude.

12. Arrangements have been made to determine the magnitudes of φ Persei with modern photometric apparatus throughout a whole period of velocity change at the next favourable opportunity.

PHASE 2
33
60
82
95
111
3800
3900
4000
4100
4200
4300
44
45
46
47
48
49

13. Two plates accompany the paper showing :—
    (16) A comparison with α Cygni (ionised metallic lines) and γ Orionis (helium lines) ; and
    (17) The spectrum of ϕ Persei at six different phases of velocity in the line of sight.

Description of the Plates.

Plate 16. Comparison of the spectrum of ϕ Persei with α Cygni and γ Orionis showing the bright Fe^+ and Ti^+ and the dark Ca^+ lines in relation to α Cygni, and the dark helium lines in relation to γ Orionis. (Spectra widened by the clepsydra apparatus.)

Plate 17. A series of photographs of ϕ Persei exhibiting the spectrum at different phases marked on the right-hand side. They correspond to negatives 536, 550, 570, 584, 590, and 515, of which the line intensities are given in Table V. and illustrated in fig. 2. Notice the strength and sharpness of $\lambda\lambda 4471{\cdot}6$ and $4026{\cdot}3$ at phase 33 days, and the strong central absorption of the hydrogen lines and the strength of the calcium line at $\lambda 3933{\cdot}8$ at phase 111 days. (Spectra widened by the clepsydra apparatus.)

On Some Measures of the Solar Rotation at Different Levels in the Chromosphere. By J. Evershed, F.R.S.

Photographs of solar spectra with iron-arc comparison were obtained in Kodaikanal in 1922 and 1923, using a prism spectrograph. This gives a dispersion in the H and K region of nearly 2 mm. per Angstrom with excellent definition. With this instrument, and a 6-inch solar image, spectra were obtained at the centre of the disc, near the polar limbs, and near the equator east and west. These were primarily for the purpose of measuring the displacements sun – arc of the iron lines at the centre and at the limb ; but the spectra of the equatorial limbs may be used also for determining the solar rotation.

A similar series was obtained for the D lines, using a potash salt in the arc, which gave narrow and very brilliant sodium lines. The grating spectrograph was used for this series.

The mean displacements of pairs of plates taken at nearly the same heliographic latitude east and west give values of the solar rotation. These are set out in Table I., Series 1 and 2, after slight corrections to reduce them to the equator and to sidereal velocity.

In Series 3 of Table I. the results are given of measures of some grating plates taken in the year 1918 of lines in the red region, near to and including the H_α line. These are solar spectra only, and were taken for the purpose of determining the rotation displacements with the greatest possible accuracy of measurement. The east and west spectra are photographed simultaneously on the same plate.

All of these series of spectra were measured by the positive-on-negative method. The probable errors are derived from the accordance of the plates in a series : the errors of measurement are much smaller.

In Table II. the various values are arranged according to the estimated heights of the absorbing gases above the photosphere; and to these I have added a value obtained from the H and K lines in sixty-one prominences photographed on the east and west limbs in the years 1908–1911. The measures were made at a mean height of 46″ above the photosphere and at mean latitude 31°. The reduction to the equator was made on the assumption of uniform angular motion. The third column of this table gives the mean deviations from plate to plate in km./sec. It is a common experience in spectrographic measures of the rotation that these deviations greatly exceed the errors of measurement, and they appear to be due to real changes of velocity. The fourth column gives the approximate values of the solar radius in the region of absorption (or emission in the case of prominences): these are expressed in seconds of arc to be added to the mean radius of the photosphere. The approximate amounts to be added are assumed to be one-half the height to which the gases ascend in the chromosphere, and are derived from measures of my eclipse plates of 1898 and 1900. The last column gives the daily angular motion, taking account of the increasing radius.

Two results of interest appear from these measures. First, the angular rotation speed increases markedly with the height of the absorbing region, as was found by Adams in his measures of the H_α line, and the line $\lambda 4227$ of calcium.* Secondly, the deviations from plate to plate in a series tend to increase also with the height.

The measures of the D lines are slightly discordant in giving a lower value than that given by the strong lines in the ultra-violet; but owing to the great variation from plate to plate, and the small number of plates, the rotation value given has little weight. Comparing the strong and the weak lines in the same plates of the H and K region, the difference is striking, the strong lines giving a distinctly larger value than the weak lines, and also a larger deviation from plate to plate. The strongest lines of all, viz. H and K, could only be measured on two plates which happened to give measurable images of the calcium-arc lines: these give the highest angular speed of all, except the prominences. When we come to the prominence region, outside the chromosphere, the deviations from plate to plate are enormous, the mean exceeding 1 km./sec. The prominences selected for the measures are those of a more or less permanent type, which may be observed successively on the east and west limbs after intervals of fourteen days. They are not usually associated with rapid movements in the line of sight.

The value of the rotation period derived from the prominences is, of course, subject to considerable revision. If it gives a fair estimate of the rotation speed, it would appear that the shortest period of the sun's rotation, about twenty-one days, occurs entirely outside the limits of the chromosphere. This outer coronal region is apparently subject to violent conflicting movements, made evident by the erratic nature of the line-shifts, even when so-called "quiescent" types of

* *An Investigation of the Rotation Period of the Sun*, by W. S. Adams, published by the Carnegie Institute of Washington, 1911.

prominences are selected for the measures. It would seem that this turbulent motion extends downward through the chromosphere with diminishing amplitude, giving rise to the disconcerting variations in spectrographic measures of the rotation.

TABLE I.

Mean Sidereal Velocities at Sun's Equator deduced from Lines of Different Intensity and Different Spectral Regions.

	Km./sec.	
Series 1. H *and* K *Region.*—Twelve plates photographed between the dates 1922 Dec. 15 and 1923 Jan. 6, all within 5° of the Sun's equator—		
H and K lines	2·126	(2 plates only).
11 strong lines of Fe and Al, 3886 to 3969, mean intensity 9·2 . .	1·950	±·024
8 weak lines of Fe, 3885 to 3977, mean intensity 4·3	1·931	±·015
Series 2. D_1 *and* D_2 *in Sun and Arc.*—Twelve plates photographed between 1922 Mar. 24 and 31, all within 5° of the equator . .	1·939	±·062
Series 3. H_α *Region.*—Nineteen plates photographed between 1918 Feb. 6 and Mar. 15, all within 12° of the equator—		
H_α line	2·067	±·028
5 Fe lines and 5 Ca lines, mean intensity 5·7	1·932	±·013

TABLE II.

Rotation Values at Varying Heights above Photosphere.

	Km./sec.	Mean Deviation.	Radius.	Angular Motion.
Weak lines in U.V. . .	1·931	·034	$962''+\frac{1}{2}''$	13°·6
Fe and Ca lines in Red .	1·932	·060	$962+\frac{1}{2}$	13·6
Strong lines in U.V. . .	1·950	·073	962+1	13·8
D lines of Sodium . .	1·939	·155	962+2	13·7
Hα (Hydrogen) . .	2·067	·128	962+6	14·7
H and K (Calcium) . .	2·126	..	962+7	15·0
Prominences . . .	2·48	1·18	962+46	16·8

The Stellar Magnitude Scales of the Astrographic Catalogue. Provisional General Survey. By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. In 1909 a simple proposal was made (*M.N.*, **69**, 392) for the comparison of the stellar magnitude scales of the Astrographic Catalogue, viz. :

> *That the number of images recorded under each unit of the magnitude scale be counted and tabulated.*

Since that time a good deal of work has been done in this way, partly at Oxford, partly by kind volunteer help, and partly by the co-operation of the Directors of participating observatories. Much still remains to be done ; but it is now possible to collect the results of 40 zones in different parts of the sky, and to make a preliminary survey of this material in respect of the distribution of the stars through the heavens.

2. Most previous discussions of this nature have related chiefly to the effect of galactic latitude without distinction between different longitudes. The early papers of this series suggested that the differential effect of galactic latitude as between bright and faint stars had been overestimated, and that another phenomenon, provisionally identified as an obscuration effect, had been underestimated.

Chapman and Melotte also (*Mem. R.A.S.*, **60**) found that the galactic differential effect was much smaller than Kapteyn's estimate. But as the collection of material proceeded, it was found that the galactic effect in the " obscured " and in the " unobscured " patches was quite definite, and of about the amount assigned by Kapteyn (*M.N.*, **75**, 608 ; 1915 June). At the same time emphasis was laid on the additional phenomenon of obscuration, owing to which

> the portion in which faint stars relatively fail is in and near the Galaxy itself (*ibid.*, p. 605).

3. Two years later, F. H. Seares made a more definite determination of the differential galactic condensation (*Ap. J.*, 1917 September) from the then published material. He apparently obtained a good first approximation which was forthwith utilised to clear further results from this condensation. Moreover, those already published have been cleared in the same way, as occasion offered. The standard process is described in *M.N.*, **78**, 55–57 : it converts log Nγ in any galactic latitude γ to log N_0, the corresponding value on the galactic equator : N representing the total number of stars per square degree brighter than a limit specified by the particular scale in use at the particular observatory.

4. The main purpose of this whole work is to co-ordinate as far as possible the magnitude scales of the different zones with one another and with a standard scale. But for a general review these individual scales may be ignored and the stars may be grouped simply by the values of log N_0. It is assumed, in fact, that at any particular observatory the stars designated (say) 10·5 or D in any zone are of the same magnitude throughout that zone. Though this may be only roughly true, it must be the basis of *any* use of the zone unless the

whole work is done over again in detail. What precisely that magnitude is may be put aside for the present purpose, which is to concentrate attention on the distribution in the zone. The extreme case is that of the limiting magnitude for the plate, *i.e.* that of the faintest stars measured: and in Table I. the results for "All" stars are presented, to show the consistency of the material. They are simply collected from the different papers already given, and are the coefficients of the first four harmonics representing the residuals of log N_0, after correction for galactic latitude. The θ of the current reductions is really R.A. and has been replaced by α.

TABLE I.

"All" Stars.

δ.	log N_0.	sin α.	cos α.	sin 2α.	cos 2α.	sin 3α.	cos 3α.	sin 4α.	cos 4α.
+64°	2·20	−·032	−·009	−·119	−·019	−·012	−·030	+·052	+·004
+63	2·17	−·005	−·025	−·035	−·033	+·003	−·012	+·015	+·028
+62	2·22	+·003	+·003	+·033	+·038	−·026	+·113	+·011	+·038
+61	2·23	−·036	+·023	+·021	−·022	+·062	+·067	+·069	+·028
+60	2·23	+·003	+·042	−·013	−·055	+·034	+·028	+·038	+·023
Means	2·21	−·013	+·007	−·023	−·018	+·012	+·033	+·037	+·024
+31	2·33	−·221	−·079	−·078	+·027	−·024	+·028	+·083	−·003
+30	2·39	−·273	−·097	−·108	−·037	−·003	−·009	+·068	−·008
+29	2·36	−·080	−·062	−·080	+·046	+·068	+·037	+·074	−·033
+28	2·38	−·097	−·156	−·140	+·068	−·003	+·024	+·038	+·040
+27	2·26	−·118	−·045	−·138	+·057	+·073	+·013	+·075	+·020
Means	2·34	−·158	−·088	−·109	+·032	+·022	+·022	+·068	+·003
+26	2·33	−·157	−·054	−·083	+·004	+·037	+·026	+·042	+·051
+25	2·39	−·074	−·038	−·052	+·041	−·043	+·101	+·065	+·024
+24	2·26	−·149	+·013	−·088	−·013	−·036	+·040	+·020	+·036
+23	2·28	−·038	+·041	−·157	−·031	−·019	+·066	+·047	+·052
+22	2·31	−·065	−·012	−·126	−·016	−·049	+·012	−·011	+·023
Means	2·31	−·097	−·010	−·101	−·003	−·022	+·049	+·033	+·037
+17	2·19	−·084	−·088	−·091	+·041	−·040	+·062	+·034	+·044
+16	2·08	−·053	−·028	−·123	+·027	−·006	+·079	+·030	+·037
+15	2·05	+·013	+·004	−·090	+·030	−·021	+·100	+·034	−·003
+14	2·16	+·012	−·027	−·086	−·073	−·106	+·062	+·012	+·028
Means	2·12	−·028	−·035	−·098	+·006	−·043	+·076	+·028	+·027
− 1	2·10	+·120	−·040	−·062	+·059	−·119	+·001	+·046	−·092
− 2	2·11	+·157	+·007	−·054	+·126	−·110	+·009	+·024	−·088
− 3	2·28	+·029	+·033	−·078	+·069	−·084	−·010	+·028	−·077
− 5	2·18	+·048	+·013	−·048	+·070	−·113	+·006	+·043	−·053
Means	2·17	+·089	+·003	−·061	+·081	−·107	+·002	+·035	−·078

TABLE I.—*continued.*

δ.	log N_δ.	sin a.	cos a.	sin $2a$.	cos $2a$.	sin $3a$.	cos $3a$.	sin $4a$.	cos $4a$.
−15°	2·27	+·050	+·053	−·128	+·024	−·029	+·014	+·118	−·031
−16	2·26	−·015	+·026	−·171	+·028	−·021	+·011	+·102	−·008
−17	2·31	−·030	+·042	−·003	+·074	−·061	+·016	+·073	−·022
−18	2·32	+·036	+·028	+·006	+·089	+·039	−·034	−·016	+·012
−19	2·27	+·130	+·104	−·052	+·026	−·031	−·032	+·042	−·009
−20	2·40	+·014	+·042	−·175	+·060	−·015	−·104	+·036	−·030
Means	2·31	+·031	+·049	−·087	+·050	−·020	−·022	+·059	−·015
−25	2·33	+·066	+·054	−·083	+·008	−·011	−·043	+·063	−·030
−27	2·18	+·023	+·086	−·048	− 063	−·002	+·041	+·084	−·005
−29	2·20	−·023	−·006	+·029	−·027	−·018	−·049	+·022	+·018
Means	2·24	+·022	+ 045	− 034	−·027	−·010	−·017	+·056	−·006
−31	2·28	−·119	−·051	−·010	+·044	+·024	−·022	+·046	+·031
−32	2·25	−·088	−·043	−·024	−·016	+·068	+·032	+·062	+·032
−34	2·27	−·054	−·022	−·065	−·072	−·013	−·019	+·048	−·001
−36	2·36	−·089	+·088	−·068	+·047	+·032	−·012	+·044	+·007
Means	2·29	−·087	−·007	−·042	+·001	+·028	−·005	+·050	+·017
−38	2·37	−·083	+·052	−·028	−·003	+·035	−·009	+·057	+·008
−41	2·37	− 024	−·022	−·032	−·016	−·001	+·021	+·006	+·027
−42	2·41	−·083	+·025	−·048	−·156	+·046	−·023	+·017	+·029
−43	2·38	−·007	+·015	−·057	−·107	+·027	−·025	+·053	+·031
Means	2·38	−·049	+·018	−·041	−·071	+·027	−·009	+·033	+·024

5. The mean value of log N_0 is 2·26, corresponding to magnitude 13·35 on the scale of Chapman and Melotte (*Mem. R.A.S.*, **60**, 152; zone ±1°). Similar tables have been made for values of log N_0 near 1·90 (mag. 12·1), 1·50 (mag. 11·3), 1·00 (mag. 10·2), 0·50 (mag. 9·1), and 0·00 (mag. 8·0): though the material becomes more affected by accidental errors as we proceed to the brighter (and fewer) stars. It seems unnecessary to repeat all these figures at the present stage; for though they all show some features of interest, the single case given above is sufficiently illustrative of them all.

6. Attention may first be directed to the remarkable consistency of the coefficient of sin $4a$. Of the 40 independent values above tabulated, only 2 are negative and these are small (−·011 and −·016). The mean value is +·045 and the mean deviation about ±·020. The values of the coefficient of cos $4a$ are also fairly consistent, though there is a notable exception near − 3° which reduces the mean value to +·004. Inspection of the other figures shows that the coefficient of sin $4a$ increases from bright stars to faint, thus representing a feature which may have been overlooked in studies of the brighter stars. Its reality

is supported by the evidence that the fourth *spherical* harmonics (of which the terms in 4a are only one form) are generally larger than the first and third. But the second is also large. We see that the coefficient of sin 2a is chiefly negative ; there are only 4 positive values, and the mean of the 40 is −·069. The mean value of cos 2a is again small—only +·006.

7. We will now collect the means of the nine belts above represented : adding similar means for the other values of log N_0.

TABLE II.

(The unit is 0·001 in log N_0.)

Dec.	log N_0.	sin a.	cos a.	sin 2a.	cos 2a.	sin 3a.	cos 3a.	sin 4a.	cos 4a.
+62°·0	9·94	− 45	+348	− 7	+ 2	+ 8	+19	+17	−34
+62·0	0·45	− 49	+193	+ 1	+ 21	+ 35	+ 3	−22	−18
+62·0	0·98	− 21	+ 69	− 10	+ 10	+ 36	+ 6	− 3	−17
+62·0	1·44	− 37	+ 31	− 22	− 3	+ 32	+25	+ 3	−11
+62·0	1·95	− 31	+ 18	− 16	− 16	+ 19	+36	+28	+ 7
+62·0	2·21	− 13	+ 7	− 23	− 18	+ 12	+33	+37	+24
Mean	1·41	− 30	+ 64	− 14	− 1	+ 27	+21	+ 9	− 3
+29·0	0·14	− 66	− 3	− 50	+ 12	+ 34	−25	+69	+49
+29·0	0·64	− 90	− 48	− 50	+ 18	+ 43	− 9	+73	+16
+29·0	1·09	−106	− 74	− 66	+ 19	+ 48	− 5	+69	+ 4
+29·0	1·59	−125	−108	− 81	+ 36	+ 51	+ 7	+73	+ 1
+29·0	1·98	−119	−113	− 81	+ 39	+ 29	+22	+82	− 8
+29·0	2·34	−158	− 88	−109	+ 32	+ 22	+22	+68	+ 3
Mean	1·53	−120	− 86	− 77	+ 29	+ 39	+ 7	+73	+ 3
+25·5	9·88	− 23	− 65	− 31	− 47	− 26	−47	+67	−17
+24·0	0·70	− 31	− 48	− 31	+ 25	− 16	+ 2	+29	− 7
+24·0	1·04	− 36	− 63	− 41	+ 31	− 23	+33	+22	+ 3
+24·0	1·47	− 81	− 74	− 50	+ 45	− 13	+57	+29	+ 7
+24·0	2·01	−111	− 25	− 72	+ 41	− 25	+53	+32	+20
+24·0	2·31	− 97	− 10	−101	− 3	− 22	+49	+33	+37
Mean	1·51	− 71	− 44	− 59	+ 28	− 20	+39	+29	+12
+15·5	9·50	− 66	− 67	+ 32	−127	+136	+65	+59	+57
+15·5	0·16	− 37	− 57	− 14	− 27	+ 99	+56	+52	+56
+15·5	0·93	+ 13	− 72	− 59	+ 18	+ 23	+52	+ 4	+24
+15·5	1·58	0	− 47	− 80	+ 30	− 7	+70	+ 8	+22
+15·5	1·93	− 11	− 43	− 94	+ 20	+ 14	+69	+23	+20
+15·5	2·12	− 28	− 35	− 98	+ 6	− 43	+76	+28	+27
Mean	1·34	− 13	− 51	− 69	+ 9	+ 17	+65	+23	+30

TABLE II.—*continued.*

Dec.	log N_0.	sin α.	cos α.	sin 2α.	cos 2α.	sin 3α.	cos 3α.	sin 4α.	cos 4α.
− 1·5	9·96	+130	− 62	− 34	−104	− 52	−73	+17	+19
− 2·7	0·71	+ 78	− 59	− 58	+ 7	− 81	−20	−30	−13
− 3·3	1·02	+105	− 34	− 32	+ 49	− 62	−12	+ 4	−29
− 3·3	1·52	+ 77	− 39	− 44	+ 90	− 75	+ 5	+ 7	−62
− 2·8	1·98	+ 70	− 25	− 44	+111	− 87	+ 8	+25	−84
− 2·8	2·17	+ 89	+ 3	− 61	+ 81	−107	+ 2	+35	−78
Mean	1·48	+ 84	− 31	− 48	+ 68	− 82	− 3	+ 8	−53
−17·5	0·63	+ 69	+ 91	− 13	+ 71	− 24	−30	−16	− 1
−17·5	0·70	+ 74	+ 82	− 20	+ 73	− 19	−35	−16	− 4
−17·5	1·42	+ 52	+111	− 36	+ 76	− 40	−36	+12	0
−17·5	1·83	+ 51	+116	− 52	+ 72	− 41	−25	+29	− 6
−17·5	2·15	+ 30	+ 82	− 83	+ 54	− 32	−20	+49	−15
−17·5	2·31	+ 31	+ 49	− 87	+ 50	− 20	−22	+59	−15
Mean	1·68	+ 48	+ 88	− 56	+ 65	− 30	−28	+29	− 8
−27·0	9·60	+ 25	− 18	− 17	−138	− 51	+16	+33	+36
−27·0	0·15	+ 5	− 30	− 57	−113	− 67	+10	−18	+31
−27·0	0·81	+ 28	+ 6	+ 4	− 98	− 16	− 8	+30	+18
−27·0	1·47	+ 32	+ 11	− 33	− 3	− 22	− 7	+37	+12
−27·0	2·05	− 38	+ 38	− 37	− 39	− 13	−24	+65	+13
−27·0	2·24	+ 22	+ 45	− 34	− 27	− 10	−17	+56	− 6
Mean	1·34	+ 10	+ 14	− 31	− 56	− 26	− 9	+34	+14
−33·3	9·75	+ 4	− 93	− 21	−166	+ 15	+57	−18	+77
−33·3	0·33	− 9	− 38	− 36	−122	+ 52	+ 3	−26	+65
−33·3	0·93	− 17	+ 5	− 20	− 27	+ 60	+ 8	+20	+54
−33·3	1·50	− 42	+ 18	− 30	− 7	+ 45	−10	+29	+40
−33·3	2·02	− 59	+ 5	− 35	+ 5	+ 41	− 9	+36	+25
−33·3	2·29	− 87	− 7	− 42	+ 1	+ 28	− 5	+50	+17
Mean	1·41	− 43	− 3	− 33	− 30	+ 45	− 3	+22	+40
−41·0	0·20	+ 85	− 2	− 67	− 60	− 5	+55	+55	−19
−41·0	0·68	+ 87	+ 27	− 76	− 56	+ 21	+43	+26	+19
−41·0	0·84	+ 86	+ 38	− 77	− 45	+ 36	+34	+23	+19
−41·0	1·60	+ 42	+ 13	− 62	− 60	+ 21	+23	+33	+12
−41·0	1·97	− 7	+ 51	− 81	− 74	+ 24	− 4	+33	+23
−41·0	2·38	− 49	+ 18	− 41	− 71	+ 27	− 9	+33	+24
Mean	1·49	+ 32	+ 29	− 67	− 61	+ 26	+17	+30	+19

8. The next step is to represent the means for each zone by a series of spherical harmonics. We might do this separately for each value of log N_0; but at the present stage it seems sufficient to represent in this way:

(*a*) The mean values for the mean value of log N_0: excluding the top row in each zone, as affected by larger accidental errors than the others, the means of the other five rows have been taken at the foot of each set.

(*b*) The rate of change of this mean with log N_0. In previous papers the rough process has been adopted of subtracting the top two rows from the bottom two, and dividing by the interval in log N_0. But a more legitimate method has now been used, weighting the differences by the interval in log N_0. The mean increases thus derived are given first, as they are required to correct the mean values to a standard value of log N_0, which has been taken as 1·50.

TABLE III.

Rates of Increase of the Coefficients for 1·00 *in Log* N_0.

(The unit is 0·001 in log N_0.)

Dec.	sin *a*.	cos *a*.	sin 2*a*.	cos 2*a*.	sin 3*a*.	cos 3*a*.	sin 4*a*.	cos 4*a*.
+62°·0	+13	−93	−12	−23	−14	+20	+33	+24
+29·0	−33	−28	−30	+11	−14	+21	+ 1	− 9
+24·0	−51	+28	−40	−10	− 3	+27	+ 5	+25
+15·5	− 2	+19	−52	+17	−75	+15	− 7	−16
− 3·0	+10	+31	− 5	+55	−20	+17	+38	−49
−17·5	−27	−24	−46	−17	+ 1	+11	+48	− 9
−27·0	− 8	+33	− 1	+43	+21	−13	+33	−14
−33·3	−40	+12	− 6	+55	−14	− 7	+33	−25
−41·0	−81	− 7	+13	−14	− 1	−32	+ 6	+ 3

9. There is a systematic appearance about most of these columns, and in correcting the mean values to log N_0=1·50 it is a question whether to use these rates of increase as they stand, or whether to combine them into any systematic law, even by a graphical process. The former process is, however, simpler and quite defensible, and has been adopted.

Thus the mean value of log N_0 for zone +62° shown in Table II. is 1·41. To correct to the standard value 1·50 we multiply the top row of Table III. by 0·09 and add to the mean values of Table II.

[TABLE IV.

TABLE IV.

Mean Values of the Coefficients for Log $N_0 = 1{\cdot}50$.

(The unit is 0·001 in log N_0.)

Dec.	sin a.	cos a.	sin $2a$.	cos $2a$.	sin $3a$.	cos $3a$.	sin $4a$.	cos $4a$.
+62°·0	− 31	+ 56	− 15	− 3	+26	+ 23	+ 12	− 1
+29·0	−119	− 85	− 76	+29	+39	+ 6	+ 73	+ 3
+24·0	− 70	− 44	− 59	+28	−20	+ 39	+ 29	+12
+15·5	− 13	− 48	− 77	+12	+ 5	+ 68	+ 22	+27
− 3·0	+ 84	− 31	− 48	+69	−82	− 3	+ 8	−53
−17·5	+ 53	+ 92	− 48	+68	−30	− 30	+ 20	− 6
−27·0	+ 9	+ 19	− 31	−49	−23	− 11	+ 39	+12
−33·3	− 47	− 2	− 34	−25	+44	− 4	+ 25	+38
−41·0	+ 33	+ 29	− 67	−61	+26	+ 17	+ 30	+19
N. Sum	−233	−121	−227	+66	+50	+136	+136	+41
S. Sum	+132	+107	−228	+ 2	−65	− 31	+122	+10

10. At the foot of the table are given the sums of the values for N. declinations and for S. declinations separately. It will be seen that for odd multiples of a these are of opposite signs, and for even multiples of the same sign. When we combine the various series of coefficients into spherical harmonics, this means that harmonics of odd index are small and of even index more prominent.

11. The functions of δ which combine with the above functions of a to make spherical harmonics have been tabulated for the first four multiples of a in *M.N.*, **72**, 213, using multiples of δ; but in practice it has been found after all more convenient to use powers of sin δ and cos δ. Further, some of the numerical coefficients (which are chiefly a matter of convenience) have been revised. Finally, the series has been extended to the sixth harmonics, for reasons which will presently appear. The coefficients are given in Table V.: where $s = \sin\delta$ and $c = \cos\delta$.

TABLE V.

a	(1). c	(2). $2sc$	(3). $4c-5c^3$	(4). $s(4c-7c^3)$	(5). $8c-28c^3+21c^5$	(6). $s(8c-36c^3+33c^5)$
$2a$	..	c^2	$3sc^2$	$6c^2-7c^4$	$s(6c^2-9c^4)$	$16c^2-48c^4+33c^6$
$3a$	..	..	c^3	$3sc^3$	$8c^3-9c^5$	$s(16c^3-22c^5)$
$4a$	..	..	..	c^4	$4sc^4$	$10c^4-11c^6$
$5a$	..	..	..	..	c^5	$5sc^5$
$6a$	..	..	..	..	..	c^6

12. Let us consider first the coefficients of sin $4a$ and cos $4a$. If we limit ourselves to the first four harmonics, the only available coefficient is c^4 or $\cos^4\delta$. Using the process of least squares to obtain its numerical factors, so that the best possible value may be secured, the result is

$$(+{\cdot}041 \sin 4a + {\cdot}002 \cos 4a)\cos^4\delta.$$

The values of the terms in sin 4α and cos 4α are shown under the heading C in Table VI.

TABLE VI.

δ.	$x_4 = \cos^4\delta$.	sin 4α. C.	sin 4α. O.	sin 4α. O−C.	cos 4α. C.	cos 4α. O.	cos 4α. O−C.	$z_4 = 10c^4 - 11c^6$.
+62°·0	+ ·05	+ 2	+ 12	+10	0	− 1	− 1	+ ·37
+29·0	+ ·59	+ 24	+ 73	+49	+1	+ 3	+ 2	+ ·93
+24·0	+ ·70	+ 29	+ 29	0	+1	+12	+11	+ ·57
+15·5	+ ·86	+ 35	+ 22	−13	+2	+27	+25	− ·19
− 3·0	+1·00	+ 41	+ 8	−33	+2	−53	−55	− ·97
−17·5	+ ·83	+ 34	+ 20	−14	+2	− 6	− 8	− ·01
−27·0	+ ·63	+ 26	+ 39	+13	+1	+12	+11	+ ·79
−33·3	+ ·49	+ 20	+ 25	+ 5	+1	+38	+37	+1·13
−41·0	+ ·33	+ 13	+ 30	+17	+1	+19	+18	+1·22
N. Sum	+2·20	+ 90	+136	+46	+4	+41	+37	
S. Sum	+3·28	+134	+122	−12	+7	+10	+ 3	

13. Now there is a systematic run about the residuals O−C: in both sin 4α and cos 4α they are negative in the middle and positive at the ends. It is clear that the inclusion of the harmonic of the fifth order could not help us, for it would change sign with the declination: but the sixth harmonic, $z_4 = 10\cos^4\delta - 11\cos^6\delta$, the values of which are shown in the last column of Table VI., is of the right nature. Forming, then, two equations for X and Z (to multiply x_4 and z_4) we get

$$X = +\cdot 028 \quad Z = +\cdot 027 \text{ for } \sin 4\alpha$$
$$X = -\cdot 004 \quad Z = +\cdot 020 \text{ for } \cos 4\alpha$$

restoring the decimal notation; thus the sixth harmonic is at least as important as the fourth.

The effects of the two terms are shown in Table VII.

TABLE VII.

Fourth and Sixth Harmonics in sin 4α *and* cos 4α.

Zone.	sin 4α. x_4.	sin 4α. z_4.	sin 4α. Sum.	sin 4α. O.	sin 4α. O−C.	cos 4α. x_4.	cos 4α. z_4.	cos 4α. Sum.	cos 4α. O.	cos 4α. O−C.
+62°·0	+ 1	+10	+11	+ 12	+ 1	0	+ 7	+ 7	− 1	− 8
+29·0	+16	+25	+41	+ 73	+32	−2	+19	+17	+ 3	− 14
+24·0	+20	+15	+35	+ 29	− 6	−3	+11	+ 8	+ 12	+ 4
+15·5	+24	− 5	+19	+ 22	+ 3	−3	− 4	− 7	+ 27	+ 34
− 3·0	+28	−26	+ 2	+ 8	+ 6	−4	−19	−23	− 53	− 30
−17·5	+23	0	+23	+ 20	− 3	−3	0	− 3	− 6	− 3
−27·0	+18	+21	+39	+ 39	0	−2	+16	+14	+ 12	− 2
−33·3	+14	+31	+45	+ 25	−20	−2	+23	+21	+ 38	+ 17
−41·0	+ 9	+33	+42	+ 30	−12	−1	+25	+24	+ 19	− 5
± Sum				±258	±83				±171	±117

14. At the foot of the table are given the numerical sums of the residuals. In sin 4a the values O are all of one sign, and we might expect the sum +258 to be considerably reduced by the $\cos^4\delta$ term in Table VI., which is also all of one sign. But the numerical sum of O−C in Table VI. is ±154—not a very great reduction. The use of the sixth harmonic reduces this to ± 83, which is a more considerable reduction. Indeed, mere inspection of the original column O shows that the factor x_4 cannot go far in explaining it, since the smallest value +8 occurs just where x is +1·00. In cos 4a there is also a considerable gain from the sixth harmonic. Indeed, the fourth alone only reduces the O from ±171 to ±168, which is practically no reduction at all.

15. This case has been given in some detail because it is representative. It has similarly been found in the other cases that the first four harmonics will not suffice, and the fifth and sixth must be taken in. This was shown by preliminary trials with the fourth alone, which did not work: but these need not all be given; we will repeat the procedure for the terms in sin 3a and cos 3a to show that it is not exceptional: for the others, we may proceed at once to the use of the harmonics of higher orders. These results imply, of course, that we ought similarly to have formed the coefficients of sin 5a, cos 5a; sin 6a, cos 6a; and this must be done. But apparently we can afford to stop there. The residuals of Table VI. suggest that higher harmonics must be small, for the remaining large residuals have an accidental appearance, though doubtless they could be reduced by including more terms. If it becomes necessary to try this, groups must probably be formed of shorter length than 1 h. in R.A.

16. Coming now to the coefficients of sin 3a and cos 3a, if we restrict ourselves to the fourth harmonic, they may be multiplied by either of the two functions of δ,

$$x_3=\cos^3\delta \quad \text{or} \quad y_3=3\sin\delta\cos^3\delta,$$

which are exhibited in the second and third columns of Table VIII.

TABLE VIII.

δ.	δ Factors.		sin 3a.			cos 3a.			z_3.
	x_3.	y_3.	C.	O.	O−C.	C.	O.	O−C.	
+62°·0	+ ·10	+ ·27	+ 5	+26	+21	+ 10	+ 23	+13	+ ·62
+29·0	+ ·67	+ ·97	+17	+39	+22	+ 42	+ 6	−36	+ ·74
+24·0	+ ·76	+ ·93	+16	−20	−36	+ 43	+ 39	− 4	+ ·37
+15·5	+ ·90	+ ·71	+12	+ 5	− 7	+ 40	+ 68	+28	− ·32
− 3·0	+1·00	− ·16	− 3	−82	−79	+ 16	− 3	−19	− ·98
−17·5	+ ·87	− ·78	−13	−30	−17	− 5	− 30	−25	− ·17
−27·0	+ ·71	− ·96	−16	−23	− 7	− 13	− 11	+ 2	+ ·62
−33·3	+ ·58	− ·96	−16	+44	+60	− 16	− 4	+12	+1·00
−41·0	+ ·43	− ·85	−14	+26	+40	− 16	+ 17	+33	+1·24
	+2·43	+2·88	+50	+50	0	+135	+136	+ 1	
	+3·59	−3·71	−62	−65	− 3	− 34	− 31	+ 3	

The sums for N. and S. declinations are given at the foot of the columns, and it will be seen that they provide us with the equations

$$+2{\cdot}43X+2{\cdot}88Y=+50 \text{ (sin } 3a) \text{ or } +136 \text{ (cos } 3a)$$
$$+3{\cdot}59X-3{\cdot}71Y=-65 \text{ (sin } 3a) \text{ or } -31 \cos 3a,$$

the solutions of which are

$$X=\cdot000,\ Y=+\cdot017 \text{ for sin } 3a\ ;\quad X=+\cdot021,\ Y=+\cdot029 \text{ for cos } 3a,$$

which have been used in forming the columns C in Table VIII.

17. But the representation is far from satisfactory. The numerical sums of the columns O are ±295 and ±201, which are only reduced by subtraction of the columns C to ±289 and ±172. Both columns O C show the feature of being negative in the middle and positive at the ends: so that they can be, at any rate partly, represented by the fifth harmonic, $z_3=8\cos^3\delta-9\cos^5\delta$, given in the last column. But after applying this term to the residuals it was found that the coefficients of x_3 and y_3 required modification, so that it is better to start again with equations for X, Y, Z, the numerical factors of x_3, y_3, and z_3. Taking the equations in three groups, we get

$$+1{\cdot}53X+2{\cdot}17Y+1{\cdot}73Z=+45 \text{ or } +68$$
$$+2{\cdot}77X-0{\cdot}23Y-1{\cdot}47Z=-107 \text{ or } +35$$
$$+1{\cdot}72X-2{\cdot}77Y+2{\cdot}86Z=+47 \text{ or } +2.$$

18. The solution of these gives

$$\left.\begin{matrix}X=-\cdot020\\ Y=+\cdot007\\ Z=+\cdot035\end{matrix}\right\} \text{ for sin } 3a \quad \text{and} \quad \left.\begin{matrix}X=+\cdot017\\ Y=+\cdot015\\ Z=+\cdot005\end{matrix}\right\} \text{ for cos } 3a.$$

The solution for cos $3a$ differs sensibly from that just given, although the Z term is small. On inspection it will be seen that this difference comes from the regrouping, which allows the zones $\delta=+15^\circ{\cdot}5$ and $\delta=-17^\circ{\cdot}5$ to neutralise each other, instead of contributing to the Y term. As we have seen, this large Y term does not really satisfy the individual results, and the new solution is no better in this respect than the old.

19. But on trial of the sixth harmonic

$$u_3=s(16c^3-22c^5)$$

we find that $c=-\cdot025u_3$ reduces the numerical sum of the residuals sensibly. Other coefficients can be found for x_3, y_3, and z_3, but their application does not materially reduce this sum, and it seems better to leave the solution as simple as possible provided it is effective.

TABLE IX.

Revised Solution for Terms in 3α.

δ.	sin 3α.					cos 3α.		
	−·020 x_3.	+·007 y_3.	+·035 z_3.	O.	O−C.	−·025 u_3.	O.	O−C.
+62°·0	− 2	+2	+22	+ 26	+ 4	−26	+ 23	+ 49
+29·0	−13	+7	+26	+ 39	+ 19	+ 7	+ 6	− 1
+24·0	−15	+7	+13	− 20	− 25	+18	+ 39	+ 21
+15·5	−18	+5	−11	+ 5	+ 29	+26	+ 68	+ 42
− 3·0	−20	−1	−34	− 82	− 27	− 9	− 3	+ 6
−17·5	−17	−6	− 6	− 30	− 1	−26	− 30	− 4
−27·0	−14	−7	+22	23	− 24	−12	− 11	+ 1
−33·3	−12	−7	+35	+ 44	+ 28	0	− 4	− 4
−41·0	− 9	−6	+43	+ 26	− 2	+25	+ 17	− 8
				±295	±159		±201	±136

20. We now come to the terms in 2α, and we may as well prepare for a solution to the sixth harmonic from the outset. The functions of δ shown in Table V. are tabulated in columns 2 to 6 of Table X. under the headings x_2, y_2, z_2, u_2, v_2.

TABLE X.

Terms in 2α.

δ.	x_2.	y_2.	z_2.	u_2.	v_2.	sin 2α.			cos 2α.		
						O.	C(x).	O−C.	O.	C.	O−C.
+62°·0	+ ·22	+ ·58	+ ·98	+·78	+1·55	− 15	−15	0	− 3	−21	+ 18
+29·0	+ ·77	+1·11	+ ·49	−·33	−1·10	− 76	−53	− 23	+ 29	+21	+ 8
+24·0	+ ·84	+1·02	+ ·13	−·51	− ·89	− 59	−57	− 2	+ 28	+36	− 8
+15·5	+ ·93	+ ·75	− ·47	−·58	− ·10	− 77	−63	− 14	+ 12	+57	− 45
− 3·0	+1·00	− ·16	− ·98	+·15	+ ·96	− 48	−69	+ 21	+ 69	+61	+ 8
−17·5	+ ·91	− ·82	− ·34	+·60	− ·31	− 48	−62	+ 14	+ 68	+21	+ 47
−27·0	+ ·79	−1·08	+ ·35	+·41	−1·00	− 31	−54	+ 23	− 49	−16	− 33
−33·3	+ ·70	−1·15	+ ·77	+·11	−1·00	− 34	−48	+ 14	− 25	−36	+ 11
−41·0	+ ·57	−1·12	+1·15	−·32	− ·37	− 67	−39	− 28	− 61	−54	− 7
						−455		±139	±344		±185

21. But at this stage it is not necessary to use all these terms or to find their coefficients by any elaborate process. We see at once that the most important term in sin 2α is that in x_2, the coefficient of which was found by a summary process to be ·069. Applying this in the column C(x) the numerical sum of residuals is at once reduced from

455 to ±139, and it seems unlikely that the addition of other terms will better this appreciably. Inspection suggests that the coefficient of u_2 may be sensible; and it was found to be +·037: but on applying it, the residuals for $\delta=+62°$ and $\delta=+24°$ were sensibly increased, so that the numerical sum only fell from ±139 to ±121. Hence for the present we may be content with the single term in x_2. It may be added that this conclusion was confirmed by solving for a number of terms, not quite by least squares, but by a shorter process nearly equivalent to it.

22. On the other hand, the terms in cos 2α require at least three terms, say

$$+\cdot025x_2, \quad +\cdot020y_2, \quad -\cdot040z_2,$$

to give any reasonable reduction of the residuals. The value of this expression is given in column C, and the O−C residuals are reduced from ±344 to ±185 only. Tests were made for the u and v terms without any satisfactory result. It is, however, on the whole, reassuring to find that these higher harmonics are not wanted for the simpler terms in 2α.

23. Coming now to the terms in sin α and cos α, there are six possible coefficients in Table V., which we may denote by the symbols x_1, y_1, z_1, u_1, v_1, w_1. These were all tested, but the details need scarcely be given. The outcome is that there are two principal terms for sin α, viz.:

$$C=-\cdot060y_1+\cdot100v_1=-\cdot060\sin 2\delta+\cdot050\,(8c-28c^3+21c^5)$$

and one principal term for cos α, viz.:

$$C=+\cdot080u_1=+\cdot080s(4c-7c^3).$$

The effect of these is shown in Table XI.

TABLE XI.

Terms in α.

δ.	sin α.				cos α.		
	O.	$-\cdot060y_1$.	$+\cdot050v_1$.	O−C.	O.	$+\cdot080u_1$.	O−C.
+62°·0	− 31	−50	+69	− 50	+ 56	+82	− 26
+29·0	−119	−51	−50	− 18	− 85	−46	− 39
+24·0	− 70	−44	−34	+ 8	− 44	−55	+ 11
+15·5	− 13	−31	+ 5	+ 13	− 48	−51	+ 3
− 3·0	+ 84	+ 6	+47	+ 31	− 31	+12	− 43
−17·5	+ 53	+34	− 5	+ 24	+ 92	+54	+ 38
−27·0	+ 9	+49	−44	+ 4	+ 19	+50	− 31
−33·3	− 47	+55	−54	− 48	− 2	+33	− 35
−41·0	+ 33	+59	−42	+ 16	+ 29	− 1	+ 30
	±459			±212	±406		±256

24. The fifth and sixth harmonics are apt to appear, therefore, in several places, and we must be prepared for the coefficients of sin 5a

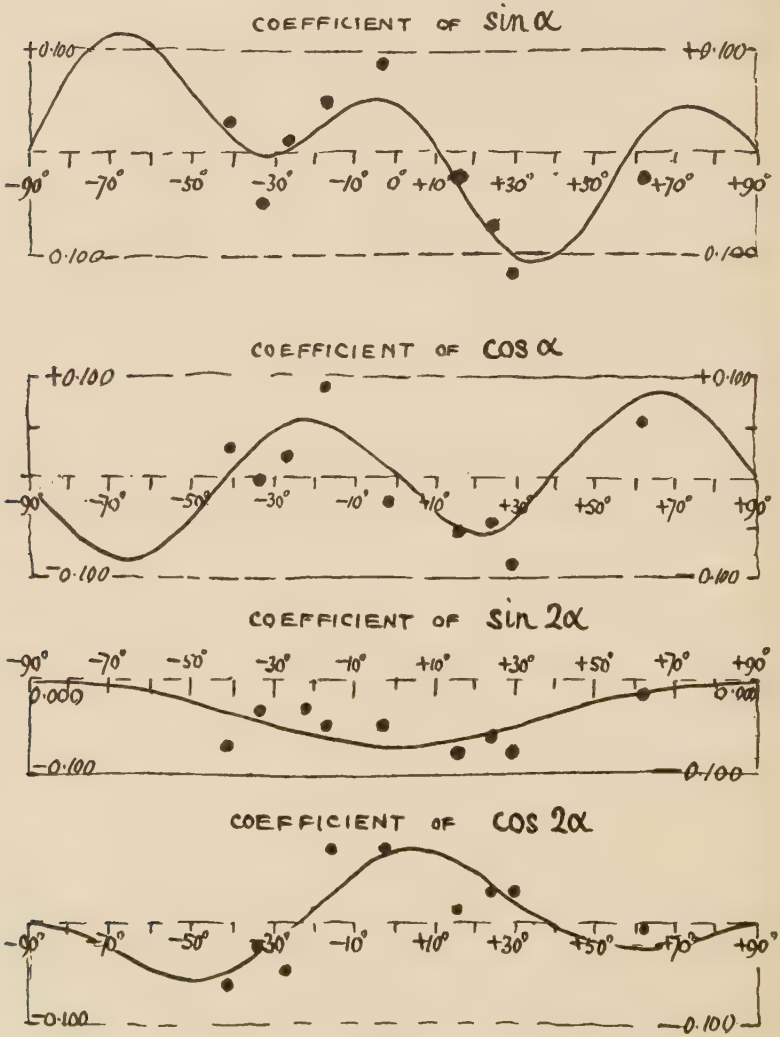


FIG. 1.

and sin 6a being sensibly systematic. Reference was therefore made back to the original figures for twenty-four hours, and these twenty-four terms analysed for terms in 5a and 6a. This entailed a good deal of work, but as the result was chiefly negative (*i.e.* no *large* terms in 5a and 6a appeared) the details need not be given at this stage.

25. Collecting the results, we find the following important terms, adopting rounded figures for the coefficients :—

$$(-\cdot060y_1+\cdot050v_1)\sin\alpha+\cdot080u_1\cos\alpha,$$
where $y_1=\sin 2\delta=2sc$, $u_1=s(4c-7c^3)$, $v_1=(8c-28c^3+21c^5)$.

$$(-\cdot070x_2)\sin 2\alpha+(\cdot025x_2+\cdot020y_2-\cdot040z_2)\cos 2\alpha,$$
where $x_2=c^2$, $y_2=3sc^2$, $z_2=6c^2-7c^4$,

$$(-\cdot020x_3+\cdot035z_3)\sin 3\alpha-\cdot025u_3\cos 3\alpha,$$
where $x_3=c^3$, $z_3=8c^3-9c^5$, $u_3=s(16c^3-22c^5)$.

$$(+\cdot030x_4+\cdot030z_4)\sin 4\alpha+\cdot020z_4\cos 4\alpha,$$
where $x_4=c^4$, $z_4=10c^4-11c^5$.

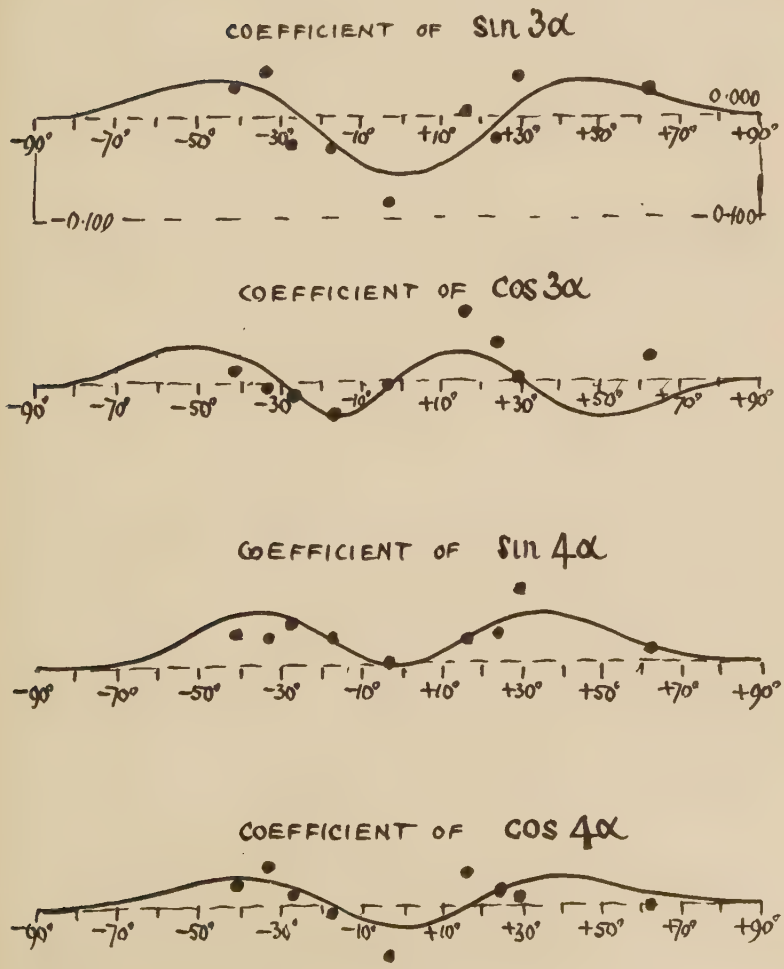


FIG. 2.

26. The comparison of these formulæ with the nine zone-groups is shown disgrammatically. It remains to be seen whether counts still to be made will fall in with these empirical curves. At present it would be premature to attempt physical interpretations. Moreover, the rates of change from bright stars to faint, though provisionally tabulated in Table III., have not yet been critically examined. But it is submitted that the results here presented suggest at least the possibility that there are systematic features of stellar distribution not hitherto realised ; that they can profitably be explored by use of the magnitude determinations of the Astrographic Catalogue in the simple manner adopted ; and that it would require a vast amount of work to explore them in any other way, seeing that a fairly complete survey of the sky is required to detect such inequalities as are represented (for instance) by $\sin 4\alpha$ (sample regions would have to be very numerous). If these facts are admitted, they form a strong support for the desirability of completing the Astrographic Catalogue without undue delay.

Summary.

1. The counts for 40 zones (each of 2° in declination) of the Astrographic Catalogue, which have been presented in detail, are now assembled for general preliminary discussion in nine groups, declinations $+62°$ to $-41°$. The galactic condensation has been removed to the first approximation.

2. We may conveniently consider :

(*a*) The distribution of the stars of mean magnitude 11.

(*b*) The variations of this distribution for one magnitude.

The quantities (*b*) are exhibited in Table III. of the present paper, but not critically discussed. For the present, attention is concentrated on (*a*).

3. Contrary to expectation, it is found that the third and fourth harmonics are by no means small and irregular. The coefficient of $\sin 4\alpha$, for instance, is systematically positive in all the groups. Even the sixth harmonic is sometimes required to represent the systematic results.

4. Hence it appears that the galactic condensation is by no means the only systematic feature in stellar distribution : there are others which can profitably be studied by means of these simple counts.

Nouvelles classes de solutions séculaires du problème astéroïdique de trois corps. Par W. W. Heinrich.

J'ai trouvé déjà en l'an 1919 certaines classes de nouvelles solutions du problème astéroïdique, qui ont reçu—grâce à certaines raisons analytiques—le nom de séculaires, de bifurcation.* Depuis lors j'ai étudié ces mouvements d'un autre point de vue † et plus tard je les ai généralisés pour le cas du problème général des trois corps.‡

Partons du problème restreint ; c'est-à-dire, étudions le mouvement d'un astéroïde de masse zéro sous l'action d'une planète troublante, tournant dans un cercle autour du corps central. Cette fois, on le sait bien, une seule classe de solutions périodiques existera. Ce sont les solutions de la première et seconde sorte qui se confondent ensemble dans ce cas. Pour simplifier, nous envisageons seulement les mouvements dans un plan.

Dans le cas du problème astéroïdique, où l'on suppose de même l'astéroïde de masse zéro, tandis que la planète troublante tourne dans une ellipse, tous les auteurs qui ont fait des recherches sur ce problème, G. W. Hill, Poincaré, C. W. L. Charlier, K. Schwarzschild, A. Wilkens, n'ont constaté, en examinant *les proximités des commensurabilités* des mouvements moyens, que des orbites à un périhélie fixe. La cause de cet échec avait été, que les dits auteurs obtiennent leurs résultats seulement par un procédé de dégénération en partant du problème général de trois corps.

Les orbites ainsi trouvées n'apparaissent que comme un cas particulier des solutions de la seconde sorte du problème général de trois corps.§ Le résultat essentiel de mes propres recherches était :

Outre les orbites périodiques à un périhélie fixe trouvées par Hill, etc., appartenant principalement à la seconde sorte, il existe pour chaque type de commensurabilité des mouvements moyens une multiple infinité des solutions exactes dont le périhélie est animé d'un mouvement de rotation séculaire. Ces dernières sont à peu près analogues aux solutions de la seconde sorte du problème restreint, plus exactément elles représentent leur continuation analytique. La plus importante note caractéristique de ces orbites nouvellement trouvées est qu'elles se rencontrent généralement, du moins aussi fréquemment que celles de la première sorte (à période courte). Ainsi, les conditions initiales de ces solutions périodiques sont très avantageusement distribuées dans l'espace, une circonstance qui se trouve étroitement liée à la longueur de leur période. J'ai divisé les solutions en deux classes et trois catégories—ces dernières d'après la longueur de leurs périodes.

* *Bull. astr.*, ii. série, tome ii., p. 425, fasc. vii. ; *Časopis math. a fys.*, Prague, 1920–21.

† *Comptes Rendus*, **177**, p. 752, 1923 (ii. sem.), Paris, etc.

‡ *Mémoires de la Société Royale des sciences*, Prague, 1922 ; *Publications de l'Université Charles*, Prague, 1923. *Comptes Rendus*, **180**, pp. 45–47, Paris.

§ Charlier, *Mech. d. Himmels*, **2**, 226.

Ce sont les orbites longuaevales, orbites séculaires à longue et à courte période.

On peut dire en général :

Les solutions séculaires à courte période donnent une explication suffisante pour les lacunes dans l'anneau formé par les petites planètes. En effet, c'est la vitesse énorme de leurs lignes des apsides qui décèle l'instabilité de l'équilibre. À ce point de vue j'ai entrepris des calculs très détaillés dans le but d'examiner la stabilité de l'anneau des astéroïdes et de ces lacunes. Les solutions séculaires à longue période correspondent à peu près exactément à une commensurabilité des mouvements moyens. Elles sont entièrement suffisantes pour représenter exactement les orbites des planètes critiques, au moyen des séries uniformément convergentes—et cela pour une durée infinie.

Ainsi j'ai construit l'orbite de la planète Hécube, avec une période principale de 1500 années juliennes, j'ai calculé de même l'orbite de Thule. Dans ce qui suit je vais simplifier les considérations publiées déjà dans le *Bull. astr.*, **2**, fasc. 7, 1922, p. 424 *et seq.*, en évitant complètement l'emploi de l'opération Delaunay. Pour atteindre ce but je vais appliquer une méthode qui mène à des autres classes générales de nouvelles solutions exactes, dont les classes de bifurcation trouvées dans le *Bull. astr.*, loc. cit., ne forment qu'un cas particulier.

Dans la seconde partie de la note j'ai construit les premiers termes des développements.

Les orbites nouvellement trouvées dans la note suivante diffèrent des orbites de bifurcation étudiées, *loc. cit.*, dans un point essentiel.

Choisissant une ellipse à la proximité d'une commensurabilité des mouvements moyens on constatera—on le sait bien—dans le cas du problème restreint (l'excentricité de Jupiter zéro)—une solution périodique de la première sorte au périhélie mobile (en rotation).

La vitesse de la ligne des apsides disparaît quand on néglige la masse troublante. Imaginons que le second paramètre troublant, l'excentricité de l'orbite de la planète Jupiter diffère de zéro, nous allons trouver comme continuation analytique des solutions de la première sorte—les solutions de bifurcation étudiées dans le *Bull. astr.*, loc. cit.

Les orbites ainsi trouvées n'admettent aucun accroissement de la rotation séculaire de la ligne des apsides qui disparaîtrait avec le paramètre de l'excentricité. Cependant en généralisant les conditions initiales et en poursuivant la méthode choisie dans le mémoire présent on trouvera facilement qu'aussi ces autres nouvelles classes des orbites séculaires jouissant de la dernière propriété caractéristique (c'est-à-dire admettant un tel accroissement) existent en réalité.

Donc on peut démontrer facilement l'existence d'une multiple infinité de solutions séculaires du type mentionné plus haut et cela pour chaque type de commensurabilité des mouvements moyens.

On pourra diviser ces solutions en deux classes : Si la période sidérale embrasse seulement une rotation du périhélie, il s'agit de la classe première, si la période renferme plus d'une rotation, on parle de la classe seconde.

Conformément à la longueur de leur période les orbites nouvellement trouvées pourront être divisées en :—

1. Solutions séculaires à période longue, avec une période comparable aux périodes connues de la théorie (de Stockwell-Lagrange) des perturbations séculaires des planètes principales.

2. Solutions séculaires à courte période, avec une période plus longue que cent années juliennes.

3. Solutions semiséculaires (longuaevales) avec une période plus courte que 100 années.

§ 1. Imaginons un corps central et une planète (Jupiter) tournant autour de lui dans une ellipse. Les équations du mouvement d'un astéroïde de masse zéro, soumis aux influences de ces deux corps (problème astéroïdique) seront données par

$$\frac{dx_1}{dt}=\frac{\partial F}{\partial y_1},\quad \frac{dy_1}{dt}=-\frac{\partial F}{\partial x_1},$$
$$\frac{dx_2}{dt}=\frac{\partial F}{\partial y_2},\quad \frac{dy_2}{dt}=-\frac{\partial F}{\partial x_2}.$$

$$F=k^4/2x_1^2+n'x_1-n'x_2+\mu F_1,\quad \mu F_1=k^2\mu[1/\Delta-r/r'^2\cos(\nu-\nu')]$$
$$=k^2\mu\Sigma l_{i_1,i_2,i_3}\cos(i_1y_1+i_2y_2+i_3M').$$

$$\Delta^2=r^2+r'^2-2rr'\cos(\nu-\nu'),\qquad F_0=k^4/2x_1^2+n'x_1-n'x_2.$$

$$\nu=v+\pi,\quad \nu'=v'+\pi',\qquad n'=k(1+\mu)^{\frac{1}{2}}/a'^{\frac{3}{2}},\quad n=k/a^{\frac{3}{2}}.$$

$$x_1=k\sqrt{a},\qquad y_1=M+\pi-\pi'-M'=(n-n')t+M_0-M_0'+\pi-\pi',$$

$$x_2=k\sqrt{a}(1-\sqrt{1-e^2}),\qquad y_2=-\pi+\pi'+M'=n't-\pi+\pi'+M_0',$$

$$\xi=(p+q)M'-p(M+\pi-\pi')-q(\pi-\pi'),\quad M'=n't+M_0',$$

$$\xi'=(p+q)M'-pM-p(\pi-\pi')=qM'-py_1.$$

Voici le sens des désignations employées ci-dessus, k est la constante de Gauss, a le grand demi-axe de l'astéroïde, e l'excentricité, π la longitude du périhélie, ν la longitude vraie, v la vraie anomalie, M l'anomalie moyenne, n le mouvement moyen diurne, r le rayon vecteur, p, q deux nombres positives, premiers entre eux, t le temps.

Les mêmes lettres munies d'accents appartiennent à la planète troublante. Imaginons la proximité d'une commensurabilité générale des mouvements moyens $n_0/n'=(p+q)/p$, ξ, ξ' désignent les deux arguments critiques. On doit accentuer que le temps t figure explicitement dans les arguments des termes trigonométriques de la fonction perturbatrice de notre problème. Les coordonnées ne satisfont pas l'intégrale bien connue de Jacobi, la dernière ne commence à exister qu'après l'introduction des nouvelles variables :

$$\frac{du}{dt}=1,\quad u=t+h,\quad \frac{dv}{dt}=-\frac{\partial F}{\partial u},\quad F''=F+v,\quad (h=\text{const.}).$$

Les équations du mouvement jouissant alors d'une intégrale de Jacobi s'écrivent:

$$\frac{dx_1}{dt}=\frac{\partial F''}{\partial y_1}, \quad \frac{dy_1}{dt}=-\frac{\partial F''}{\partial x_1},$$
$$\frac{dx_2}{dt}=\frac{\partial F''}{\partial y_2}, \quad \frac{dy_2}{dt}=-\frac{\partial F''}{\partial x_2},$$
$$\frac{du}{dt}=\frac{\partial F''}{\partial v}, \quad \frac{dv}{dt}=-\frac{\partial F''}{\partial u},$$

$$F''=F+v=\text{const.}$$

Nous allons poser $t=(1+\kappa)\tau$—bien que le temps figure explicitement —et nous essayons de déterminer κ convenablement. La condition que les équations du mouvement jouissent d'une période préscrite devra être satisfaite. En posant $\mu=0$, les trajectoires (dégénérées) deviennent périodiques.

Voici les valeurs des éléments pour le commencement de la période

$$x_1=x_1^0, \quad x_2=x_2^0, \quad y_1=y_1^0=M_0-M_0'+\pi-\pi', \quad y_2=y_2^0=M_0'-\pi+\pi'.$$

Pour atteindre la continuation analytique valable pour le cas $\mu \gtrless 0$ nous allons être obligés de changer légèrement les valeurs initiales en leur ajoutant des petits accroissements α_k, β_k.

Il sera :

$$x_1=x_1^0+\alpha_1, \quad x_2=x_2^0+\alpha_2, \quad y_1=y_1^0+\beta_1, \quad y_2=y_2^0+\beta_2.$$

En premier lieu on devra déterminer la période du mouvement. Dans ce qui suit, nous allons démontrer que la possibilité de l'existence d'un paramètre κ différent de zéro, forcera l'apparition de la rotation séculaire du mouvement du périhélie d'une grandeur entièrement définie.

Nous allons appeler ce paramètre κ, le paramètre du périhélie, f sera lié avec les deux mouvements moyens par les relations :

$$pn_1-(p+q)n'+qf=0, \quad n_1=n+h=(n_0+\kappa n')/(1+\kappa),$$
$$f=\kappa n'/(1+\kappa), \quad n_0/n'=(p+q)/p.$$

Ces équations sont de la même forme que celles, valables pour le problème restreint. En posant $e'=0$, il s'agit d'une période synodique des solutions de la première sorte. Au commencement et vers la fin de la période les trois corps se trouvent sur la ligne commune des apsides, cette dernière ayant tourné d'un certain angle pendant la durée de la période. La longueur de cette période synodique sera

$$T'=\frac{2\pi q}{n_1-n'}=(1+\kappa)\frac{2\pi p}{n'},$$

ce qui pourra être démontré aussi par une méthode géométrique, d'ailleurs très claire. Par contre en supposant l'étude du problème astéroïdique $e' \gtrless 0$ on sera obligé de passer à une période sidérale. Cela signifie en d'autres termes que la rotation synodique doit s'accroître à 360 degrés. On trouvera facilement les systèmes suivants des formules, fixant la période sidérale T ; p, q, r, l', s sont les nombres entiers positifs,

p, q premiers entre eux. Nous trouvons pour la première classe (avec une révolution unique de la ligne des apsides) :

$$|f|\,T'r=2\pi,\quad qr=(n_1-n')/|f|=q/p(n'/|f|\mp 1),$$
$$T'r=T,\quad n'/|f|=rp\pm 1=l',\quad r=1,\ 2,\ 3\ \ldots$$
$$T=T'r=2\pi/n'\cdot rp(rp\pm 1)/(rp\pm 1\mp 1)=\tau'_{\mathfrak{Z}}(rp\pm 1).$$

Les deux signes donnent respectivement un mouvement direct et rétrograde du périhélie. $\tau'_{\mathfrak{Z}}$ signifie la durée de révolution de la planète perturbatrice.

Pour la deuxième classe, avec plus de révolutions de la ligne des apsides on trouvera semblablement :

$$|f|\,T'r=2\pi s,\qquad pr/s\pm 1=(pr\pm s)/s=n'/|f|=\bar{l}/s,$$
$$qr/s=(n_1-n')/|f|=q/p\cdot(n'/|f|\mp 1),\qquad T'=\tau'_{\mathfrak{Z}}(pr\pm s).$$

Maintenant nous pouvons indiquer les valeurs des éléments pour la fin de la période sidérale, trouvée par la méthode précédente. Nous obtenons pour

$$\mu\gtrless 0,\quad t=(1+\kappa)2p\pi r/n'=T'r=T=(1+\kappa)T_1,\quad T_1=2\pi pr/n',$$
$$x_1=x_1^0+\alpha_1+\phi_1,\quad y_1=y_1^0+(n_0-n')T+\psi_1,$$
$$x_2=x_2^0+\alpha_2+\phi_2,\quad y_2=y_2^0+n'T+\psi_2,$$

où les ϕ, ψ signifient les soi-disant déviations finales, données par les équations suivantes :

$$\frac{d\phi_1}{d\tau}=(1+\kappa)\frac{\partial F}{\partial y_1},\quad \frac{d\psi_1}{d\tau}=-n_0+n'-(1+\kappa)\frac{\partial F}{\partial x_1},$$
$$\frac{d\phi_2}{d\tau}=(1+\kappa)\frac{\partial F}{\partial y_2},\quad \frac{d\psi_2}{d\tau}=\qquad -n'-(1+\kappa)\frac{\partial F}{\partial x_2}.$$

Si les solutions sont périodiques, les déviations finales disparaissent et nous obtenons

$$\phi_1=(1+\kappa)\int_0^{T_1}\frac{\partial F}{\partial y_1}d\tau=0,\quad -\psi_1=(1+\kappa)\int_0^{T_1}\frac{\partial F}{\partial x_1}d\tau+n_0-n'=0,$$
$$\phi_2=(1+\kappa)\int_0^{T_1}\frac{\partial F}{\partial y_2}d\tau=0,\quad -\psi_2=(1+\kappa)\int_0^{T_1}\frac{\partial F}{\partial x_2}d\tau+n'=0.$$

Donc nous avons quatre variables indépendantes pour déterminer les cinq inconnues κ, α_1, α_2, β_1, β_2.

En posant $\kappa=0$, on obtient facilement les solutions bien connues de la deuxième sorte de Hill et Poincaré, jouissant d'un périhélie fixe. Cependant dans ce qui suit nous nous intéressons seulement aux solutions séculaires, c'est pourquoi nous posons $\alpha_2=0$. Les équations en ψ mènent directement aux deux conditions :

$$0=\kappa\frac{\partial F_0}{\partial x_1}+\frac{\partial^2 F_0}{\partial x_1^2}\alpha_1+\ \ldots\ +\mu\frac{\partial[F_1]}{\partial x_1}+\ \ldots$$
$$0=\kappa\frac{\partial F_0}{\partial x_2}+\ \ldots\qquad\qquad +\mu\frac{\partial[F_1]}{\partial x_2}+\ \ldots$$

où nous avons posé pour la valeur moyenne de la fonction F, composée de la partie séculaire ordinaire et de la partie critique.

$$[F_1]=\frac{1}{T_1}\int_0^{T_1} F_1 d\tau=\Sigma D\cos(m_1\xi+m_2\xi'),$$

m_1, m_2 désignant deux nombres entiers pos. nég. et zéro.

$$\begin{aligned}\xi&=(p+q)M'-p(M+\pi-\pi')-q(\pi-\pi')=-py_1+qy_2\\&=[(p+q)n'-pn_0](1+\kappa)\tau-pM_0+(p+q)M_0'-(p+q)(\pi-\pi'),\\\xi'&=(p+q)M'-pM\qquad\qquad-p(\pi-\pi')=qM'-py_1\\&=[(p+q)n'-pn_0](1+\kappa)\tau-pM_0+(p+q)M_0'-p(\pi-\pi').\end{aligned}$$

Après avoir calculé la quadrature, tous les termes critiques disparaissent. Cette remarque se rapporte en premier lieu à tous les termes qui contiennent le temps explicitement dans les arguments des termes trigonométriques et par conséquent laissent sortir en dehors des fonctions trigonométriques le temps explicite dans les équations en ϕ et ψ données ci-dessus.

Nous allons donner plus tard quelques détails du calcul. Le deuxième membre des équations en ψ donne, en négligeant les termes des ordres plus élevés $\frac{\kappa}{1+\kappa}=\frac{f}{n'}=-\frac{\mu}{n'}\frac{\partial[F_1]}{\partial x_2}$, mais cette expression représente (nous allons le démontrer plus tard) le mouvement séculaire du périhélie—ainsi le nom " paramètre de périhélie " se trouve pleinement justifié.

La première équation en ψ_1 s'écrit après avoir employé la dernière expression et profité de l'hypothèse

$$\begin{gathered}n_0/n'=(p+q)/p,\quad n_0+dn_0=n,\quad n_0\kappa=3a_1/x_1^4+h,\\-f(p+q)+pdn_0-ph=0,\quad n_1=(n_0+\kappa n')/(1+\kappa)=n_0+3a_1/x_1^4+h=n+h=n_1,\end{gathered}$$

où nous avons posé $h=\mu\frac{\partial[F_1]}{\partial x_1}$ et interpreté la valeur moyenne de l'expression $\frac{3a_1}{x_1^4}=dn_0$ comme l'accroissement du mouvement moyen.

En additionnant à cette équation $pn_0\quad n'(p+q)=0$, nous obtenons

$$\left.\begin{aligned}(n+h)p\pm q|f|-(p+q)n'=0\\n_1p\pm q|f|-(p+q)n'=0\end{aligned}\right\}\quad\ldots\quad(I.)$$

Cette condition renferme deux paramètres. Elle donne plus particulièrement à chaque excentricité de l'orbite du planétoide la valeur moyenne du grand demie-axe correspondant, ainsi que la rotation du périhélie conjuguée.

Cependant toutes les excentricités ainsi choisies ne mènent pas aux solutions séculaires.

En premier lieu on pourra choisir seulement les valeurs qui font f (resp. la valeur conjuguée de κ) commensurables avec n', qui mènent le mouvement total vers la fin de la période sidérale à la même position absolue, non seulement à la configuration originaire.

Donc nous choisissons pour ce but seulement les valeurs de e, qui mènent à $\kappa=\pm\frac{1}{pr}$ pour la classe première et à $\kappa=\pm\frac{s}{pr}$ pour la classe deuxième. En effet on aura :

$$|f|\,\mathrm{T}'r=2\pi s,\quad \pm|f|=\kappa n'/(1+\kappa),\quad \mathrm{T}'=(1+\kappa)2\pi p/n',\quad \kappa=\pm s/pr\,;\ .\quad \text{(II.)}$$

par conséquent $\kappa=\pm\frac{s}{pr}$; *q.e.d.*

Il est à noter qu'il faut supprimer la condition (II.) dans le cas $e'=\theta$, et que la première changera en une condition pour la continuation analytique des solutions périodiques de la première sorte au voisinage d'une commensurabilité des mouvements moyens.*

Nous allons étudier les deux équations en ϕ. Nous trouvons directement :

$$\phi_1=0=\alpha_1\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial x_1\partial y_1}d\tau+\alpha_2\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial x_2\partial y_1}d\tau+\beta_1\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial y_1^{\,2}}d\tau+$$
$$+\beta_2\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial y_1\partial y_2}d\tau+\ .\ .\ .\ +\mu\int_0^{\mathrm{T}_1}\frac{\partial\mathrm{F}_1}{\partial y_1}d\tau+\ .\ .\ .$$

$$\phi_2=0=\alpha_1\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial x_1\partial y_2}d\tau+\alpha_2\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial x_2\partial y_2}d\tau+\beta_1\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial y_1\partial y_2}d\tau+$$
$$+\beta_2\mu\int_0^{\mathrm{T}_1}\frac{\partial^2\mathrm{F}_1}{\partial y_2^{\,2}}d\tau+\ .\ .\ .\ +\mu\int_0^{\mathrm{T}_1}\frac{\partial\mathrm{F}_1}{\partial y_2}d\tau+\ .\ .\ .$$

Les deux équations contiennent le facteur μ. La condition de la possibilité du prolongement analytique exige :

$$\int_0^{\mathrm{T}_1}\frac{\partial\mathrm{F}_1}{\partial y_1}d\tau=\int_0^{\mathrm{T}_1}\frac{\partial\mathrm{F}_1}{\partial y_2}d\tau=0,$$

équations qui peuvent être mises sous la forme :

$$\frac{\partial[\mathrm{F}_1]}{\partial y_1}=\frac{\partial[\mathrm{F}_1]}{\partial y_2}=0,$$

$$[\mathrm{F}_1]=\Sigma\mathrm{D}\cos(m_1\xi+m_2\xi')-\Sigma\mathrm{D}\cos[-p(m_1+m_2)y_1+qm_1y_2+qm_2\mathrm{M}']$$
$$=\Sigma\mathrm{D}\cos\{(m_1+m_2)[(p+q)n'-pn_0](1+\kappa)\tau-p(m_1+m_2)\mathrm{M}_0$$
$$+(p+q)(m_1+m_2)\mathrm{M}_0'-(\pi-\pi')[m_1(p+q)+m_2p]\}.$$

Ces deux conditions donnent :

$$\frac{\partial[\mathrm{F}_1]}{\partial y_1}=\Sigma(-p)(m_1+m_2)\mathrm{D}\sin(m_1\xi+m_2\xi')$$
$$=\Sigma(-p)(m_1+m_2)\mathrm{D}\sin[-p(m_1+m_2)y_1^{\,0}+qm_1y_2^{\,0}+q\mathrm{M}_0'm_2]=0,$$

$$\frac{\partial[\mathrm{F}_1]}{\partial y_2}=\Sigma qm_1\mathrm{D}\sin(m_1\xi+m_2\xi')$$
$$=\Sigma qm_1\mathrm{D}\sin[-p(m_1+m_2)y_1^{\,0}+qm_1y_2^{\,0}+qm_2\mathrm{M}_0']=0.$$

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* *Astr. Nachr.*, Band 192, p. 327.

Il faut accentuer expressément que les équations sont complètement indépendantes l'une de l'autre, tandis qu'il n'en serait pas ainsi dans le cas bien connu des solutions à courte période de la première sorte (du problème restreint). En effet, leur éventuelle dépendance mutuelle ne saurait être causée par l'existence de l'intégrale de Jacobi, qui n'existe pas. Les équations sont satisfaites seulement en supposant que

$$\xi=\xi_0=\xi'=\xi_0'=0, \quad y_1^0=M_0-M_0'+\pi-\pi'=0, \quad y_2^0=M_0'-\pi+\pi'=0.$$

Ces conditions mènent aux configurations initiales bien connues de la conjonction et de l'opposition symétriques, etc., aussi bien qu'aux solutions générales, comme nous allons le démontrer une fois de plus dans ce qui suit. Il sera nécessaire de donner quelques détails de ce calcul bien délicat.

Au lieu de développer les termes particuliers conformément à la théorie de Cauchy-Poincaré on peut obtenir les intégrales correspondantes implicitement et développer le résultat ensuite. En effet, la théorie mentionnée exige le développement d'un type général du terme

$$l\int_0^{T_1}\cos\,[m(1+\kappa)\tau+c]d\tau, \quad m=n_0 i+n' i', \text{ etc.},$$

suivant les puissances de κ et le calcul de quadratures des coefficients en question. Au lieu de cela nous pouvons trouver la quadrature originaire en entier et ne développer qu'ensuite. On constatera ainsi facilement que le résultat final disparait, circonstance qui ne serait ni claire ni visible autrement :

$$l\int_0^{T_1}\cos\,[m(1+\kappa)\tau+c]d\tau=\frac{-\sin\,[T_1 m(1+\kappa)\tau+c]+\sin\,c}{m(1+\kappa)}=0,$$

$T_1 m(1+\kappa)=2\pi\times$ un nombre entier (en négligeant μ^2, $\mu\kappa$).

La sommation des intégrales particulières sera toujours admissible c'est-à-dire, les deux résultats devront coïncider nécessairement parce que le développement—avant d'avoir déterminé la quadrature—est uniformément convergent, et les termes particuliers sont intégrables.

Le calcul suivant n'aura besoin d'aucune remarque explicative :

$$F_1/k^2=\Sigma l_{i_1,\,i_2,\,i_3}\cos\,(i_1y_1+i_2y_2+i_3M')=\Sigma l_{i,\,i',\,j}\cos\,[iM+i'M'+j(\pi-\pi')],$$

$$i_1=i, \quad i_2=i-j, \quad i_3=j+i',$$

$$1/k^2\frac{\partial F_1}{\partial y_l}=-\Sigma i_l l_{i_1,\,i_2,\,i_3}\sin\,(i_1y_1+i_2y_2+i_3M').$$

1. Les termes non critiques, l'argument de temps t figure explicitement, ou bien n'existe pas du tout :

$$-\frac{1}{T_1}\int_0^{T_1}\Sigma i_l l_{i_1,\,i_2,\,i_3}\sin\,[\{in_0(1+\kappa)+i'n'(1+\kappa)\}\tau+j(\pi-\pi')]d\tau$$

$$=-\sum\frac{i_l l_{i_1,\,i_2,\,i_3}\cos\,[(in_0+i'n')(1+\kappa)T_1+j(\pi-\pi')]-\cos\,j(\pi-\pi')}{T_1(in_0+i'n')(1+\kappa)}\equiv 0.$$

2. Les termes critiques du type $s\xi$ (s désigne un nombre entier quelconque) :

$$\xi=-py_1+qy_2=(p+q)\mathrm{M}'-p(\mathrm{M}+\pi-\pi')-q(\pi-\pi')$$
$$=[(p+q)n' \quad pn_0](1+\kappa)\tau-p\mathrm{M}_0+(p+q)\mathrm{M}_0'-(p+q)(\pi-\pi'),$$
$$\xi_0=-p\mathrm{M}_0+(p+q)\mathrm{M}_0'-(p+q)(\pi-\pi')=-py_1^0+qy_2^0,$$
$$l_{i_1,\,i_2,\,0}\cos s\xi=l_{-ps,\,qs,\,0}\cos s\xi=\mathrm{D}_s\cos s\xi\,;\quad \rho=\begin{cases}-p\\+q.\end{cases}$$

$$\frac{1}{\mathrm{T}_1}\int_0^{\mathrm{T}_1}d\tau\Sigma\rho s\mathrm{D}_s\sin s\xi$$
$$=\frac{1}{\mathrm{T}_1}\int_0^{\mathrm{T}_1}d\tau\Sigma(-\rho s)\mathrm{D}_s\sin s[\{(p+q)n'-pn_0\}(1+\kappa)\tau-p\mathrm{M}_0+(p+q)\mathrm{M}_0'-(p+q)(\pi-\pi')]$$
$$=\Sigma\rho s\mathrm{D}_s\sin s[p\mathrm{M}_0-(p+q)\mathrm{M}_0'+(p+q)(\pi-\pi')]\equiv 0.$$

3. Les termes critiques du type $s\xi'$:

$$\xi'=-py_1+q\mathrm{M}'=(p+q)\mathrm{M}' \quad p\mathrm{M}-p(\pi-\pi')$$
$$=[(p+q)n'-pn_0](1+\kappa)\tau-p\mathrm{M}_0+(p+q)\mathrm{M}_0'-p(\pi-\pi'),$$
$$\xi_0'=-py_1^0+q\mathrm{M}'=-p\mathrm{M}_0+(p+q)\mathrm{M}'-p(\pi-\pi'),$$
$$l_{i_1,\,0,\,i_3}\cos s\xi'=l_{-ps,\,0,\,qs}\cos s\xi'=\mathrm{D}'_s\cos s\xi',\quad \rho=\begin{cases}-p\\0.\end{cases}$$

$$\frac{1}{\mathrm{T}_1}\int_0^{\mathrm{T}_1}d\tau\Sigma(-\rho)\mathrm{D}'_s\sin s\xi'$$
$$=\frac{1}{\mathrm{T}_1}\int_0^{\mathrm{T}_1}d\tau\Sigma(-\rho s)\mathrm{D}'_s\sin s[\{(p+q)n' \quad pn_0\}(1+\kappa)\tau-p\mathrm{M}_0+(p+q)\mathrm{M}_0'-p(\pi-\pi')]$$
$$=\Sigma\rho s\mathrm{D}'_s\sin s[p\mathrm{M}_0-(p+q)\mathrm{M}_0'+p(\pi-\pi')]\equiv 0.$$

4. Les termes critiques du type $m_1\xi+m_2\xi^{\xi}$ (m_1, m_2 sont les nombres entiers) :

$$l_{i_1,\,i_2,\,i_3}=l_{-p(m_1+m_2),\,qm_1,\,qm_2}=\mathrm{D}''_{m_1,\,m_2},\quad \rho=\begin{cases}-p(m_1+m_2),\\+qm_1\end{cases}$$

$$m_1\xi+m_2\xi'=-p(m_1+m_2)y_1+qm_1y_2+m_2q\mathrm{M}'=[(p+q)n'-pn_0](1+\kappa)\tau(m_1+m_2)$$
$$-p\mathrm{M}_0(m_1+m_2)+(p+q)\mathrm{M}_0'(m_1+m_2)-(\pi-\pi')[(p+q)m_1+pm_2]$$
$$=(p+q)\mathrm{M}'(m_1+m_2)-p\mathrm{M}(m_1+m_2)-(\pi-\pi')[(p+q)m_1+pm_2].$$

$$\frac{1}{\mathrm{T}_1}\int_0^{\mathrm{T}_1}d\tau\Sigma(-\rho)\mathrm{D}''_{m_1,\,m_2}\sin(m_1\xi+m_2\xi')$$
$$=\Sigma(-\rho)\mathrm{D}''_{m_1,\,m_2}\sin[\{(p+q)n'-pn_0\}(1+\kappa)(m_1+m_2)\tau+$$
$$-p\mathrm{M}_0(m_1+m_2)+(p+q)\mathrm{M}_0'(m_1+m_2)-(\pi-\pi')[(p+q)m_1+pm_2]$$
$$=\Sigma\rho\mathrm{D}''_{m_1,\,m_2}\sin[(m_1+m_2)\{p\mathrm{M}_0-(p+q)\mathrm{M}_0'\}+(\pi-\pi')\{(p+q)m_1+pm_2\}]$$
$$=\Sigma(-\rho)\mathrm{D}''_{m_1,\,m_2}\sin[m_1qy_2^0-p(m_1+m_2)y_1^0+qm_2\mathrm{M}_0'].$$

Retournons maintenant aux conditions $\psi_1=\psi_2=\phi_1=\phi_2=0$, où nous avons posé $\alpha_2=0$. Après avoir éliminé α_1, β_1, β_2 de ces quatre équations, nous obtenons une courbe analogue à celle de Poincaré (*Méthodes*

nouvelles, p. 84), $\Phi(\kappa, \mu)=0$. On ne pourra pas affirmer que tous les points de cette courbe appartiennent à une solution périodique. Ce ne sont que certaines points satisfaisant à la condition (II.) qui représentent une solution séculaire.

On pourra dire seulement : Les solutions de $\psi_1=\psi_2=\phi_1=\phi_2=0$—à savoir α_1, β_1, β_2, κ—peuvent être développées en séries, convergentes et valables seulement pour une certaine durée du temps, d'ailleurs assez courte, non pas pour toujours.

Ce n'est que pour certains points isolés, conjugués aux κ, fixés plus haut, que les séries deviennent convergentes et périodiques. Si la période est plus courte que la durée de convergence de la solution générale non périodique, alors les séries resteront convergentes pour toujours, grâce à la répétition de la configuration initiale. Et en effet nous avons démontré dans le *Bull. astr.*, loc. cit., combien la période était relativement courte (pour les solutions séculaires à courte période ne dépassant pas la limite 1000 années juliennes, et pour les solutions longuaevales moins de 100 années).

À première vue une discontinuité semble subsister :

Il semble qu'il serait impossible de passer par une suite continue de solutions périodiques dans le but de s'approcher des mouvements dégénérés de Képler pour $\mu=0$. De plus une solution κ qui ne satisfait pas aux conditions (I.), (II.), n'étant pas sidérale—le temps t figure en dehors de signes trigonométriques (sin, cos) dans les termes de l'ordre μ^2. Ainsi il semble que les solutions fixées plus haut ne représentent pas la continuation analytique des ellipses dernières.

Bien qu'il ne soit pas nécessaire de parcourir un tel passage en poursuivant une série continue de solutions périodiques, on peut toujours tourner facilement la difficulté.

Représentons la courbe dans l'espace en employant les axes rectangulaires. Envisageons la courbe $\Phi(\kappa, \mu, e', e_0)=0$, e', e_0 sont les données de la question. Il est en effet impossible de changer continuellement κ avec μ sans gâter la commensurabilité des mouvements séculaires des périhélies. Mais en partant d'un point appartenant à une solution périodique κ, on pourra changer les deux autres paramètres e_0, e' en poursuivant une ligne, perpendiculaire au plan κ, μ—c'est-à-dire, en gardant les valeurs κ, μ fixées.

Après avoir atteint un certain e' appartenant à une solution de bifurcation, ce qui sera toujours possible, on pourra changer ensuite κ avec μ continuellement. C'est ainsi qu'on pourra facilement s'approcher des mouvements de Képler (à conf. *Bull. astr.*).

La solution de bifurcation mentionnée plus haut fixera chaque fois la durée de période. Les solutions séculaires étant ainsi dispersées par l'espace κ, μ, e', je trouve le nom de " solutions séculaires, lacunaires " le plus convenable.

§ 2. Nous allons maintenant développer les premiers termes des solutions nouvellement trouvées. Nous employons les équations originaires du mouvement ainsi que la forme de la fonction perturbatrice fixée plus haut.

Les arguments de termes trigonométriques ont généralement, on le sait bien, la forme suivante :

$$i\mathrm{M}+i'\mathrm{M}'+j(\pi-\pi')=i(\mathrm{M}+\pi-\pi'-\mathrm{M}')+(i-j)(\mathrm{M}'-\pi+\pi')+(j+i')\mathrm{M}'$$
$$=i_1y_1+i_2y_2+i_3\mathrm{M}'.$$
$$i=i_1,\quad i-j=i_2,\quad j+i'=i_3.$$

Nous allons supposer la proximité d'une commensurabilité générale des mouvements moyens. La fonction perturbatrice contiendra deux parties, la partie critique et non critique. Il sera $n_0/n'=(p+q)/p$. Les arguments critiques ont la forme :

$$i_3=0,\quad e'=0,\quad \xi=-py_1+qy_2,\quad i_1=-ps,\quad i_2=qs,\quad i_3=0,$$
$$i_2=0,\quad e'\gtrless 0,\quad \xi'=-py_1+q\mathrm{M}',\quad i_1=-ps,\quad i_2=0,\quad i_3=qs,\quad i=j.$$

En introduisant la désignation,

$$l_{i_1,\,i_2,\,0}=l_{-ps,\,qs,\,0}=\mathrm{D}_s,\quad \xi_0=-p\mathrm{M}_0+(p+q)\mathrm{M}_0'-(p+q)(\pi-\pi')=-py_1{}^0+qy_2{}^0,$$
$$l_{i_1,\,0,\,i_3}=l_{-ps,\,0,\,qs}=\mathrm{D}'_s,\quad \xi_0'=-p\mathrm{M}_0+(p+q)\mathrm{M}_0'-p(\pi-\pi')=-py_1{}^0+q\mathrm{M}_0',$$
$$l_{i_1,\,i_2,\,i_3}=l_{-p(m_1+m_2),\,qm_1,\,qm_2}=\mathrm{D}''_{m_1,\,m_2},$$
$$m_1\xi_0+m_2\xi_0'=-p(m_1+m_2)\mathrm{M}_0+(p+q)(m_1+m_2)\mathrm{M}_0'-(\pi-\pi')[pm_2+(p+q)m_1]$$
$$=-p(m_1+m_2)y_1{}^0+qm_1y_2{}^0+qm_2\mathrm{M}',$$
$$\mu\mathrm{F}_1=k^2\mu\Sigma l_{i_1,\,i_2,\,i_3}\cos(i_1y_1+i_2y_2+i_3\mathrm{M}')=\mathrm{R}+\mathrm{T}'',$$
$$\mu\mathrm{R}=k^2\mu\Sigma\mathrm{D}_s\cos s\xi_0+k^2\mu\Sigma\mathrm{D}'_s\cos s\xi_0'+k^2\mu\Sigma\mathrm{D}''_{m_1,\,m_2}\cos(m_1\xi_0+m_2\xi_0')$$

nous obtenons

$$\mathrm{F}_1=\mathrm{R}+\mathrm{T}''.$$

Pour une raison qui va devenir claire plus tard, nous séparons expressément les termes des arguments particuliers. La somme première contenue dans R ne renferme pas e' du tout.

$$\mu\mathrm{T}''=k^2\mu\Sigma' l_{i_1,\,i_2,\,i_3}\cos(i_1y_1+i_2y_2+i_3\mathrm{M}').$$

L'accent de Σ' veut dire que dans la somme T″ tous les termes critiques sont à supprimer. Dans les deuxièmes membres des équations du mouvement, du moins dans les parties contenant le facteur μ, on met d'ordinaire les valeurs osculatrices des éléments. Nous allons y mettre au lieu de cela—les valeurs temporaires moyennes. C'est un procédé introduit par Poisson dans sa théorie de la Lune. Nous désignons ces valeurs par a, e, n_1, h, f, $\bar{y}_1$, $\bar{y}_2$. On obtiendra leur expression analytique au cours du calcul. Enfin au lieu des éléments a, n_1, nous pouvons employer aussi bien ceux appartenant à la commensurabilité exacte a_0, n_0 où

$$n_0/n'=(p+q)/p,\quad n_1=n+h=(n_0+\kappa n')/(1+\kappa)=n_0+3a_1/x_1{}^4+h,$$
$$n_0=k/a_0{}^{\frac{3}{2}},\qquad n=n_0+3a_1/x_1{}^4=k/a^{\frac{3}{2}}.$$

La proximité de la commensurabilité exige $q\bar{y}_2=p\bar{y}_1$; $\bar{y}_1$, $\bar{y}_2$ dénotant respectivement les valeurs moyennes :

$$q\bar{y}_2=p\bar{y}_1,\quad i_1\bar{y}_1+i\bar{y}_2=\left(i_1\frac{q}{p}+i_2\right)\bar{y}_2,\quad \bar{y}_1=n+h-n',\quad \bar{y}_2=n'-f.$$

Les équations du mouvement s'écrivent alors : puisque

$$\frac{\partial F}{\partial y_1}=\frac{\partial F}{\partial y_1^0},\quad \frac{\partial F}{\partial y_2}=\frac{\partial F}{\partial y_2^0},\quad y_1^0=M_0-M_0'+\pi-\pi',\quad y_2^0=M_0'-\pi+\pi',$$

$$\frac{dx_1}{dt}=k^2\mu\Sigma ps D_s\sin s\xi_0+k^2\mu\Sigma ps D'_s\sin s\xi_0'$$
$$+k^2\mu\Sigma p(m_1+m_2)D''_{m_1,m_2}\sin(m_1\xi_0+m_2\xi_0')$$
$$-k^2\mu\Sigma' i_1 l_{i_1,i_2,i_3}\sin[\bar y_2(i_1q/p+i_2)t+i_3n't+i_1y_1^0+i_2y_2^0+i_3M_0'],$$

$$\frac{dx_2}{dt}=-k^2\mu\Sigma qs D_s\sin s\xi_0\qquad -k^2\mu\Sigma m_1 q D''_{m_1,m_2}\sin(m_1\xi_0+m_2\xi_0')$$
$$-k^2\mu\Sigma' i_2 l_{i_1,i_2,i_3}\sin[\bar y_2(i_1q/p+i_2)t+i_3n't+i_1y_1^0+i_2y_2^0+i_3M_0'],$$

$$\frac{dy_1}{dt}=k^4/x_1^3-n'-k^2\mu\sum\frac{\partial D_s}{\partial x_1}\cos s\xi_0-k^2\mu\sum\frac{\partial D'_s}{\partial x_1}\cos s\xi_0'$$
$$-k^2\mu\sum\frac{\partial D''_{m_1,m_2}}{\partial x_1}\cos(m_1\xi_0+m_2\xi_0')-k^2\mu\sum{}'\frac{\partial l_{i_1,i_2,i_3}}{\partial x_1}\cos[\bar y_2(i_1q/p+i_2)t$$
$$+i_3n't+i_1y_1^0+i_2y_2^0+i_3M_0'],$$

$$\frac{dy_2}{dt}=n'-k^2\mu\sum\frac{\partial D_s}{\partial x_2}\cos s\xi_0-k^2\mu\sum\frac{\partial D'_s}{\partial x_2}\cos s\xi_0'$$
$$-k^2\mu\sum\frac{\partial D''_{m_1,m_2}}{\partial x_2}\cos(m_1\xi_0+m_2\xi_0')-k^2\mu\sum{}'\frac{\partial l_{i_1,i_2,i_3}}{\partial x_2}\cos[\bar y_2(i_1q/p+i_2)t$$
$$+i_3n't+i_1y_1^0+i_2y_2^0+i_3M_0'].$$

La première, deuxième et quatrième équation s'intègrent par une simple quadrature, la troisième équation pourra être traitée (intégrée) de la même manière après y avoir mis les résultats tirés de l'équation première.

Nous allons obtenir ainsi :

$$x_1=x_1^0+tk^2\mu\Sigma ps D_s\sin s\xi_0+tk^2\mu\Sigma ps D'_s\sin s\xi_0'$$
$$+tk^2\mu\Sigma p(m_1+m_2)D''_{m_1,m_2}\sin(m_1\xi_0+m_2\xi_0')$$
$$+k^2\mu\sum{}'\frac{i_1 l_{i_1,i_2,i_3}}{\bar y_2(i_1q/p+i_2)+i_3n'}\cos[\bar y_2(i_1q/p+i_2)t+i_3n't+i_1y_1^0+i_2y_2^0+i_3M_0'],$$

$$x_2=x_2^0-tk^2\mu\Sigma qs D_s\sin s\xi_0-tk^2\mu\Sigma m_1 q D''_{m_1,m_2}\sin(m_1\xi_0+m_2\xi_0')$$
$$+k^2\mu\sum{}'\frac{i_2 l_{i_1,i_2,i_3}}{\bar y_2(i_1q/p+i_2)+i_3n'}\cos[\bar y_2(i_1q/p+i_2)t+i_3n't+i_1y_1^0+i_2y_2^0+i_3M_0'],$$

$$y_1=y_1^0+t\left[\frac{k^4}{x_1^{0^3}}\quad n'-k^2\mu\sum\frac{\partial D_s}{\partial x_1}\cos s\xi_0-k^2\mu\sum\frac{\partial D'_s}{\partial x_1}\cos s\xi_0'\right.$$
$$\left.-k^2\mu\sum\frac{\partial D''_{m_1,m_2}}{\partial x_1}\cos(m_1\xi_0+m_2\xi_0')\right]-\tfrac{3}{2}t^2k^6\mu/x_1^{0^4}\{\Sigma ps D_s\sin s\xi_0$$
$$+\Sigma ps D'_s\sin s\xi_0'+\Sigma p(m_1+m_2)D''_{m_1,m_2}\sin(m_1\xi_0+m_2\xi_0')\}$$
$$-k^2\mu\sum{}'\left\{\frac{3k^4}{x_1^{0^4}}\frac{i_1 l_{i_1,i_2,i_3}}{[(i_1q/p+i_2)\bar y_2+i_3n']^2}+\frac{1}{(i_1q/p+i_2)\bar y_2+i_3n'}\frac{\partial l_{i_1,i_2,i_3}}{\partial x_1}\right\}$$
$$\sin[\bar y_2 t(i_1q/p+i_2)+i_3n't+i_1y_1^0+i_2y_2^0+i_3M_0'],$$

$$y_2=y_2{}^0+t\Big[n'-k^2\mu\sum\frac{\partial D_s}{\partial x_2}\cos s\xi_0-k^2\mu\sum\frac{\partial D'_s}{\partial x_2}\cos s\xi_0'$$
$$-k^2\mu\sum\frac{\partial D''_{m_1,m_2}}{\partial x_2}\cos(m_1\xi_0+m_2\xi_0')\Big]-k^2\mu\sum{}'\frac{1}{\bar{y}_2(i_1q/p+i_2)+i_3n'}\frac{\partial l_{i_1,i_2,i_3}}{\partial x_2}$$
$$\sin[\bar{y}_2t(i_1q/p+i_2)+i_3n't+i_1y_1{}^0+i_2y_2{}^0+i_3M_0'],$$

$$x_1{}^0=k\sqrt{a}=k\sqrt{a_0}+\alpha_1,\quad x_2{}^0=k\sqrt{a}(1-\sqrt{1-e^2})=k\sqrt{a_0}(1-\sqrt{1-e_0{}^2})+\alpha_2,$$

où a, e, désignent les valeurs temporaires moyennes. Par cette méthode nous avons obtenu approximativement les intégrales générales des équations du mouvement. Maintenant on doit en déduire les expressions analytiques particulières des solutions séculaires.

Pour ce but le choix suivant sera nécessaire :

1. Les parties des expressions du grand demi-axe et de l'excentricité, multipliées par le temps t doivent disparaître. On l'atteindra en supposant : $\xi_0=\xi_0'=0, 180°$.

En calculant le temps à partir du passage de la planète perturbatrice par le périhélie, nous pouvons poser $M_0'=0$ sans restreindre la généralité, pour obtenir les deux conditions :

$$pM_0+(p+q)(\pi-\pi')=0,\quad pM_0+p(\pi-\pi')=0,$$

qui seront satisfaites par les positions initiales bien connues de la conjonction et de l'opposition symétriques :

$$pM_0=0, 180°,\quad M_0=0, \frac{180°}{p},\quad \pi-\pi'=0, 180°.$$

De même on obtiendra facilement les conditions des solutions générales qui ne correspondent pas aux conjonctions ni aux oppositions symétriques.

2. Les expressions analytiques des angles pourront renfermer, en général, les termes proportionels au temps, cependant nous devons supposer que ces derniers s'accroissent pendant la période sidérale d'un multiple de 360°.

Nous obtenons, en même temps, le sens géométrique de tous les deux deuxièmes membres de nos équations pour y_1 et y_2,

$$\bar{y}_1=k^4/x_1{}^{03}-n'-k^2\mu\sum\frac{\partial D_s}{\partial x_1}\cos s\xi_0-k^2\mu\sum\frac{\partial D'_s}{\partial x_1}\cos s\xi_0'$$
$$-k^2\mu\sum\frac{\partial D''_{m_1,m_2}}{\partial x_1}\cos(m_1\xi_0+m_2\xi_0')$$

signifie le mouvement moyen troublé en longitude,

$$h=-k^2\mu\sum\frac{\partial D_s}{\partial x_1}\cos s\xi_0-k^2\mu\sum\frac{\partial D'_s}{\partial x_1}\cos s\xi_0'$$
$$-k^2\mu\sum\frac{\partial D''_{m_1,m_2}}{\partial x_1}\cos(m_1\xi_0+m_2\xi_0')$$

la perturbation du mouvement moyen en longitude,

$$\bar{y}_2 = n' - k^2\mu \sum \frac{\partial D_s}{\partial x_2} \cos s\xi_0 - k^2\mu \sum \frac{\partial D'_s}{\partial x_2} \cos s\xi_0' - k^2\mu \sum \frac{\partial D''_{m_1,m_2}}{\partial x_2} \cos (m_1\xi_0 + m_2\xi_0')$$

donne le mouvement du périhélie rapporté au système des axes tournants, nous obtenons ainsi le mouvement de périhélie :

$$f = \kappa n'/(1+\kappa) = k^2\mu \sum \frac{\partial D_s}{\partial x_2} \cos s\xi_0 + k^2\mu \sum \frac{\partial D'_s}{\partial x_2} \cos s\xi_0' + k^2\mu \sum \frac{\partial D''_{m_1,m_2}}{\delta x_2} \cos (m_1\xi_0 + m_2\xi_0').$$

En supposant $2\pi qr$, resp. $2\pi pr$ pour les accroissements des angles y_1, y_2 vers la fin de la période sidérale $T = T'r$, nous obtenons les deux conditions :

$$2\pi qr = T\bar{y}_1, \quad 2\pi pr = T\bar{y}_2,$$

exprimant que, pour l'existence des solutions séculaires, seront admissibles seulement les valeurs de $\bar{y}_1$, $\bar{y}_2$ telles que $\bar{y}_2 q = \bar{y}_1 p$.

Il est évident que cette équation coïncide avec la condition obtenue déjà auparavant. Et en effet nous n'avons qu'à y mettre les expressions tirées des formules données plus haut, pour obtenir :

$$(n'-f)q = p(n+h-n'), \quad p(n+h) - (p+q)n' + qf = 0 \quad . \quad . \quad \text{(I.)}$$

L'équation contient deux paramètres a, e. Après avoir choisi l'excentricité arbitrairement nous obtenons au moyen de notre relation le a conjugué resp. n et ensuite f appartenant à cela. Mais toutes les valeurs ainsi obtenues ne mènent pas aux solutions périodiques, en effet une autre condition doit être ajoutée, à savoir—il faut que la période sidérale soit un multiple de période des deuxièmes membres des équations du mouvement, parce que ces derniers contiennent le temps explicitement dans la forme $n't$. Cette condition sera satisfaite par le choix géométrique (II.) constaté expressément déjà auparavant.

Ainsi la solution séculaire du type général trouvé plus haut sera (abstraction faite des termes du second ordre de la masse troublante) une ellipse dont le moyen mouvement diffère cependant d'un petit accroissement de celui qui se déduirait d'après les lois de Képler. Cette ellipse tourne en outre dans son plan avec une vitesse de la ligne des apsides f, dans le sens direct ou rétrograde conformément au signe de f.

Les valeurs moyennes du grand demi-axe et de l'excentricité de cette ellipse en rotation sont liées par les deux relations (I.) et (II.). Les deux équations (I.), (II.) sont de beaucoup plus générales que celles trouvées déjà dans le *Bull. astr.*, loc. cit., et renferment les dernières comme un cas particulier. La chose est évidente déjà par le fait que dans les équations plus générales (I.), (II.) l'excentricité de la planète troublante figure explicitement (dans la formule analytique pour f). Nous allons accentuer en particulier le fait que les solutions séculaires générales ainsi fixées admettent aussi une autre rotation séculaire de périhélie—disparaissant avec e'—en dehors de celle appartenant aux solutions de la première sorte. L'expression de f sera donnée par

$$f=k^2\mu\sum\frac{\partial D_s}{\partial x_2}\cos s\xi_0+k^2\mu\sum\frac{\partial D'_s}{\partial x_2}\cos s\xi_0'$$
$$+k^2\mu\sum\frac{\partial D''_{m_1,m_2}}{\partial x_2}\cos(m_1\xi_0+m_2\xi'_0).$$

Pour obtenir les solutions particulières de bifurcation étudiées dans le *Bull. astr.*, loc. cit., il faudra ajouter les deux conditions :

$$a_1\sum\frac{\partial^2 D_s(x_1^0,x_2^0)}{\partial x_1^2}\cos s\xi_0+a_2\sum\frac{\partial^2 D_s(x_1^0,x_2^0)}{\partial x_1\,\partial x_2}\cos s\xi_0+\ \ldots$$
$$+\sum\frac{\partial D'_s(x_1^0+a_1,x_2^0+a_2)}{\partial x_1}\cos s\xi_0'$$
$$+\sum\frac{\partial D''_{m_1,m_2}(x_1^0+a_1,x_2^0+a_2)}{\partial x_1}\cos(m_1\xi_0+m_2\xi_0')=0,$$
$$a_1\sum\frac{\partial^2 D_s(x_1^0,x_2^0)}{\partial x_1\,\partial x_2}\cos s\xi_0+a_2\sum\frac{\partial^2 D_s(x_1^0,x_2^0)}{\partial x_2^2}\cos s\xi_0+\ \ldots$$
$$+\sum\frac{\partial D'_s(x_1^0+a_1,x_2^0+a_2)}{\partial x_2}\cos s\xi_0'$$
$$+\sum\frac{\partial D''_{m_1,m_2}(x_1^0+a_1,x_2^0+a_2)}{\partial x_2}\cos(m_1\xi_0+m_2\xi_0')=0,$$

qui représentent deux relations entre les paramètres initiaux. Il sera possible de constater semblablement aussi des mouvements, où seulement la seconde de deux parties du mouvement de périhélie existe, à savoir celle multiplée par e'.

Pour cela nous obtenons la condition :

$$\sum\frac{\partial D_s(x_1^0+a_1,x_2^0+a_2)}{\partial x_2}\cos s\xi_0+a_1\sum\frac{\partial^2 D'_s(x_1^0,x_2^0)}{\partial x_1\,\partial x_2}\cos s\xi_0'$$
$$+a_2\sum\frac{\partial^2 D'_s(x_1^0,x_2^0)}{\partial x_2^2}\cos\xi_0'+\ \ldots\ +a_1\sum\frac{\partial D''_{m_1,m_2}(x_1^0,x_2^0)}{\partial x_1\,\partial x_2}\times$$
$$\times\cos(m_1\xi_0+m_2\xi_0')+a_2\sum\frac{\partial^2 D''_{m_1,m_2}(x_1^0,x_2^0)}{\partial x_2^2}\cos(m_1\xi_0+m_2\xi_0')=0,$$

qui représente aussi cette fois une relation entre a_0, e_0, a_1, a_2. Le périhélie restera fixe mais sera soumis aux grandes oscillations dans le cas où les éléments initiaux sont liés par l'équation suivante :

$$\sum\frac{\partial D_s(x_1^0+a_1,x_2^0+a_2)}{\partial x_2}\cos s\xi_0+\sum\frac{\partial D'_s(x_1^0+a_1,x_2^0+a_2)}{\partial x_2}\cos s\xi_0'$$
$$+\sum\frac{\partial D''_{m_1,m_2}(x_1^0+a_1,x_2^0+a_2)}{\partial x_2}\cos(m_1\xi_0+m_2\xi_0')=0.$$

Dans ce cas la rotation séculaire du périhélie forcée par e' compensera le mouvement originaire de périhélie des solutions de la première sorte—qui servaient de point de départ—ou bien les deux rotations disparaissent séparément.

Observatoire de l'Université Tchèque,
Prague: 1924 *January.*

On Sir Isaac Newton's Method of Determining the Parabolic Orbit of a Comet. By Professor A. N. Kriloff, Member of the Russian Academy of Sciences.

§ 1. Sir Isaac Newton's method of determining the parabolic orbit of a comet is explained in Proposition XLI. of the third book of the *Principia* and in the Lemmas VII.–XI. preceding this proposition.

Although this method forms one of the most admirable achievements of Newton, it does not now receive the attention it deserves, and is scarcely ever mentioned in modern treatises on theoretical astronomy.

The aim of this paper is to call the attention of students of astro-

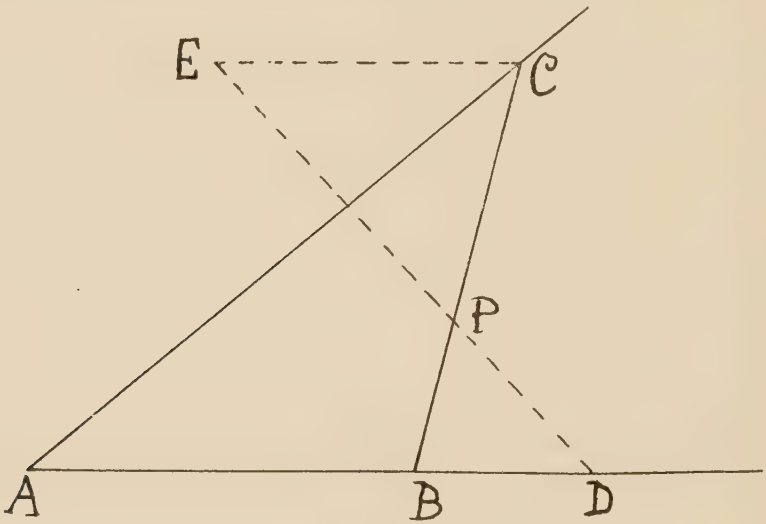

FIG. 1.

nomy to these wonderful pages of the *Principia*, and to show that they are as perfect as anything else he ever wrote.

When explaining this method I shall use modern notation and phraseology in order to avoid, for the reader, those difficulties which are not inherent in the matter itself.

§ 2. I will first explain the lemmas which form the foundation of Newton's method.

LEMMA VII.—*To draw through a point* P *a line* BC, *the parts of which,* PB *and* PC, *contained between this point and the lines* AB *and* AC, *will be in a given ratio to one another.*

Through the given point P (fig. 1) draw any line PD and produce it to such a point E that PE : PD is in the given ratio. Through E draw EC parallel to AB; the straight line PCB will be the required one.

LEMMA VIII.—*Let* ABC *be a parabola having its focus in* S. *The chord* AC *cuts off the segment* ABC, *of which* Iμ *is the diameter and* μ

the vertex. Take on Iμ *produced* $\mu O = \frac{1}{2} I\mu$, *join* OS *and produce it to* ξ, *so that* $S\xi = 2SO$. *If a comet at* B *moves on the arc* CBA, *and* ξB, *which cuts* AC *in* E, *be drawn, the point* E *will cut from the chord* AC *the part* AE, *which is very approximately proportional to the time.*

Join EO (fig. 2) intersecting the arc ABC in Y and draw at the vertex μ the tangent μX, which meets EO in X, then

$$\text{area } AEX\mu A : \text{area } ACBY\mu A = AE : AC \quad . \quad . \quad (1)$$

For

$$\text{area } AEX\mu A = AI\mu + IEX\mu$$
$$AI\mu = \tfrac{2}{3} AI \,.\, I\mu \,.\, \sin I$$
$$IEX\mu = \tfrac{1}{2} I\mu [IE + \mu X] \sin I = \tfrac{2}{3} IE \,.\, I\mu \,.\, \sin I,$$

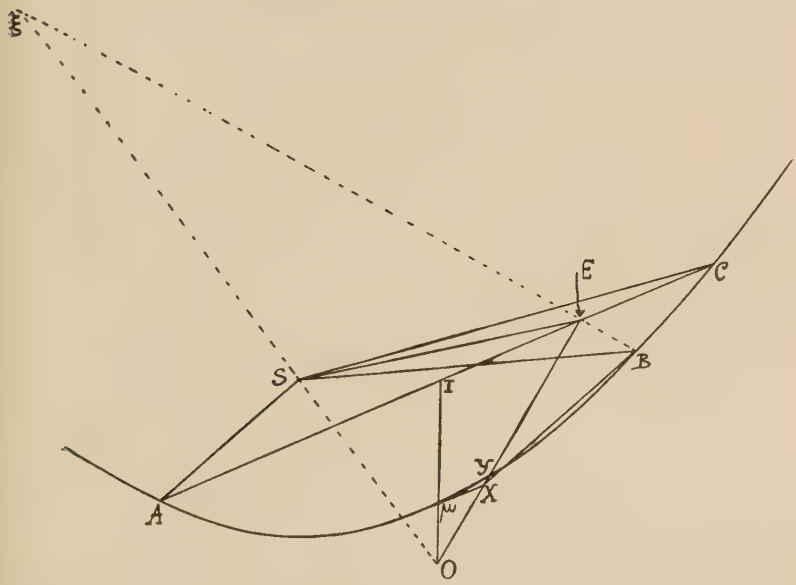

Fig. 2.

since

$$\mu X = \tfrac{1}{3} IE.$$

From these we derive

$$AEX\mu A = \tfrac{2}{3}(AI + IE) I\mu \,.\, \sin I = \tfrac{2}{3} AE \,.\, I\mu \,.\, \sin I,$$

but

$$ACBY\mu A = \tfrac{2}{3} AC \,.\, I\mu \,.\, \sin I,$$

hence the proportion (1).

The areas of the triangles ASE and ASC are obviously in the ratio AE : AC, hence we have by addition

$$ASEX\mu A : ASCBY\mu A = AE : AC \quad . \quad . \quad . \quad (2)$$

Since

$$\xi O : SO = 3 : 1$$

and

$$EO : XO = 3 : 1$$

SX is parallel to EB and

$$\text{area } SEB = \text{area } EXB,$$

hence

$$ASEX\mu A + EXB - SEB = ASEX\mu A = ASBX\mu A$$

and proportion (2) becomes

$$ASBX\mu A : ASCBY\mu A = AE : AC \quad . \quad . \quad . \quad (2')$$

but the area $ASBX\mu A$ is very nearly equal to the area of the sector $ASBY\mu A$, this equality being the more exact the nearer the point B is to the vertex μ, and the ratio of the sectors is that of the corresponding time intervals.

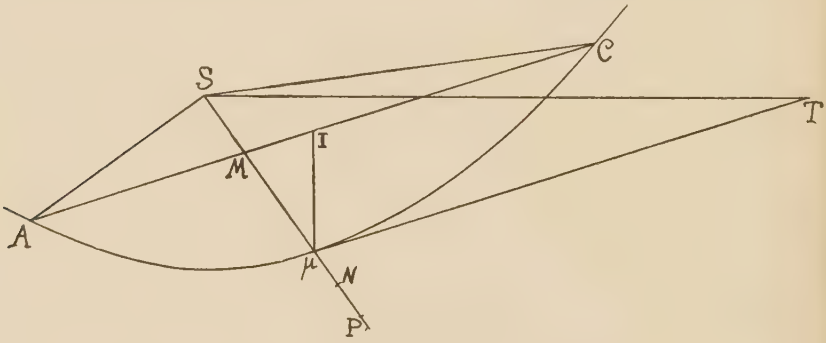

FIG. 3.

LEMMA IX.—*The lengths* $I\mu$, μM, *and* $\frac{AI^2}{4S\mu}$ *are equal to one another,* M *being the point of intersection of* AC *and* $S\mu$.

"Because the length $4S\mu$ is twice the parameter (*latus rectum*) of the parabola relating to the vertex μ."

This same property of the parabola is given in Lemma XIII. of the first book, where it is simply stated *patet ex conicis*. The first part of the lemma is obvious, because μ being the vertex of the segment $A\mu C$ (fig. 3) the tangent μT is parallel to the chord AC, and μI being a diameter this tangent bisects the angle formed by $I\mu$ and $S\mu$ produced, hence the triangle $M\mu I$ is isosceles and

$$M\mu = \mu I.$$

The distance from the vertex μ to the directrix of the parabola is the *semi-latus rectum* relating to this vertex; denoting this distance by $\frac{p}{2}$, we see by the property of the parabola that $S\mu = \frac{p}{2}$, but the equation of the parabola referred to its diameter and tangent is $y^2 = 2px$. In our case for the point A we have $y = AI$, $x = \mu I$, hence the relation $AI^2 = 4S\mu \cdot I\mu$, which is nothing else than the equation of the parabola.

Lemma X.—*If on* Sμ *produced lengths* μN *and* μP *be set off so that* $\mu N=\frac{1}{3}I\mu$ *and*

$$SP : SN=SN : S\mu,$$

SP *will be the distance from the sun* S *at which the comet has such a velocity that it would describe (with constant velocity) a distance equal to the chord* AC *in the same time as that in which it describes the arc* AμC *of its parabolic trajectory.*

If the comet were to continue its motion from μ along the tangent μT (fig. 3) during the time in which it describes the arc AμC, the length μT would be such that the area SμT=SAμC.

But we have, if Sμ and AC meet in M at the angle α,

$$\begin{aligned} SA\mu C=ASC+AC\mu&=\tfrac{1}{2}AC\,.\,SM\,.\,\sin\alpha+\tfrac{2}{3}I\mu\,.\,AC\,.\,\sin\alpha \\ &=\tfrac{1}{2}AC(SM+\tfrac{4}{3}I\mu)\sin\alpha=\tfrac{1}{2}AC\,.\,SN\,.\,\sin\alpha, \end{aligned}$$

and

$$S\mu T=\tfrac{1}{2}S\mu\,.\,\mu T\,.\,\sin\alpha,$$

hence

$$\mu T : AC=SN : S\mu.$$

In the parabolic motion the squares of the velocities are in the inverse ratio of the distances from the sun, hence with an obvious notation

$$V_\mu : V_P=SP^{\frac{1}{2}} : S\mu^{\frac{1}{2}},$$

but by construction

$$SP=\frac{SN^2}{S\mu},$$

hence

$$SP : S\mu=SN^2 : S\mu^2,$$

therefore

$$V_\mu : V_P=SN : S\mu=\mu T : AC,$$

and

$$\frac{\mu T}{V_\mu}=\frac{AC}{V_P}=\tau,$$

where τ is the time of describing the arc AμC.

In a previous communication (*M.N.*, **84**, 392) I have shown, following Lagrange, that this theorem of Newton, if expressed analytically, is nothing else than what was afterwards called the Euler-Lambert theorem, expressed by the formula

$$6\kappa\tau=(r_1+r_2+c)^{\frac{3}{2}}\mp(r_1+r_2-c)^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$SA=r_1,\ SC=r_2,\ AC=c.$$

If we put S$\mu=\zeta$ we have, as shown in the above-mentioned paper,

$$\kappa\sqrt{2\tau}=\frac{c}{\sqrt{\zeta}}\left(\zeta+\frac{c^2}{48\zeta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for the immediate expression of Newton's theorem ; κ is a constant for

the solar system, and for it Newton, in Proposition XL., gives the value

$$\kappa = 0{\cdot}01720212$$

from the astronomical data available at his time. This constant is now called the "Gaussian constant," and its value, as given by Gauss, is

$$\kappa = 0{\cdot}01720209895.$$

It may be added here, that the velocity of the comet at any distance, say SP from the sun, by this Proposition XL. is expressed by the formula

$$V_P = \frac{\kappa\sqrt{2}}{SP^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Lemma XI.—*A comet falling without initial velocity towards the sun from a distance* $SN = S\mu + \frac{1}{3}I\mu$ *under the action of a constant force, corresponding to this distance* SN, *describes a space equal to* $I\mu$ *during one-half of the time, in which it describes the arc* AC *of its orbit.*

As shown in the previous lemma, the comet during the time τ of description of the arc $A\mu C$ would describe a space equal to the chord AC in uniform motion with a velocity V_P corresponding to the distance $SP = \frac{SN^2}{S\mu}$.

At equal distances the velocity of motion in a circle is to the velocity of motion in a parabola as $1 : \sqrt{2}$, hence during the time interval τ the arc σ described in the motion in a circle of radius SP would be

$$\sigma = \frac{AC}{\sqrt{2}} = v_0\tau,$$

where v_0 is the velocity on the circle. The acceleration in this motion

$$w = \frac{v_0^2}{r} = \frac{v_0^2}{SP},$$

hence the height of fall with this acceleration during the time τ is

$$h = \tfrac{1}{2}w\tau^2 = \tfrac{1}{2}\frac{v_0^2}{r}\tau^2 = \frac{\sigma^2}{2SP} = \frac{AC^2}{4SP},$$

and during the semi-interval the height will be one-quarter of the above or

$$h_1 = \frac{AI^2}{4SP}.$$

At the distance SN the acceleration, being in the inverse ratio of the squares of the distances, is $w \cdot \frac{SP^2}{SN^2}$, hence the corresponding height h_2 will be

$$h_2 = h_1 \cdot \frac{SP^2}{SN^2} = \frac{AI^2}{4SP} \cdot \frac{SP^2}{SN^2} = \frac{AI^2}{4S\mu} = I\mu = M\mu$$

by Lemma VIII.

§ 3. Having proved these lemmas, Newton gives his graphical method for the determination of a parabolic orbit as follows :

Proposition XLI. Problem XXI.—*To determine from three given observations the orbit of a comet moving in a parabola.*

" Having attempted this most difficult (*longe difficillimum*) problem in many ways, I have," says Newton, " framed several propositions in the first book (determination of a conic satisfying given conditions) which were intended to help in the solution. Afterwards I discovered the following much simpler solution :

" Three observations, the time intervals between which are nearly equal, are to be chosen; that interval during which the comet moves more slowly being taken a little greater than the other one, so that the difference of the time intervals is to their sum approximately in the same ratio as this sum to 600 days more or less, in order that the point E (fig. 2, Lemma VIII.) may fall nearly on the point M, and, if not, is situated towards I rather than towards A. If such observations are not available, a new place of the comet is to be computed by Lemma VI." (in this lemma Newton explains his general method of interpolation).

Further, we use a slightly modernised notation and formulæ where Newton uses only words and proportions.

Let S (fig. 4) represent the sun, T_1, T_2, T_3 the positions of the earth, T_1A, T_2B, T_3C the projections on the plane of the ecliptic of the visual rays from the earth to the comet, τ_1 the time interval between the first and the second observations, τ_2 between the second and the third expressed in mean solar days, and $\tau=\tau_1+\tau_2$ the whole time interval.

Denote by X that length which the comet would describe during the time τ moving uniformly with the velocity corresponding (in parabolic motion) to the mean distance of the earth from the sun. This length is to be computed by the formula

$$X=\frac{2\pi}{365\cdot256}\,.\,a\,.\,\sqrt{2}\,.\,\tau=0\cdot01720212\,.\,a\sqrt{2}\,.\,\tau,$$

where a is the length of the said mean distance (semi-major axis of the earth's orbit), and let T_2V be the perpendicular drawn from T_2 on the chord T_1T_3.

For the determination of the orbit we proceed in the following way :—

(1) On the intermediate line T_2B a point B is taken and assumed to be the projection on the plane of the ecliptic of the position of the comet at the time of the second observation (it will be shown later how this point may be chosen). Draw towards the sun a length BE which is to T_2V in the same ratio as $SB\,.\,ST_2^2$ to the cube of the distance of the comet at this instant from the sun, hence to the quantity $(BS^2+T_2B^2\,.\,\tan^2\beta_2)^{\frac{3}{2}}$, where β_2 is the latitude of the comet at the time of the second observation.

(2) Through the point E draw the straight line AEC, the segments of which are in the ratio $\tau_1:\tau_2$; the points A and C would be approximately the projections of the positions of the comet at the first and third observations, if B had been taken in the true position corresponding to the second observation.

(3) Bisect AC in I, and erect the perpendicular Ii, where i is its

intersection with the line Bi parallel to AC. Through i draw the line Si to meet AC in the point λ, and complete the parallelogram $i\mu\lambda$I. Take the point σ on AC so that Iσ=3Iλ and draw the straight line σSξ, from which cut the length $\sigma\xi$=3Sσ+3$i\lambda$.

(4) Erase the line AC and the points E and I and draw through B in the direction of the point ξ a new length BE which is to the former one in the ratio BS^2 : (Sμ+$\frac{1}{3}i\lambda$)^2 ; through the new point so obtained draw the new line AEC, the segments of which, AE and EC, are in the ratio

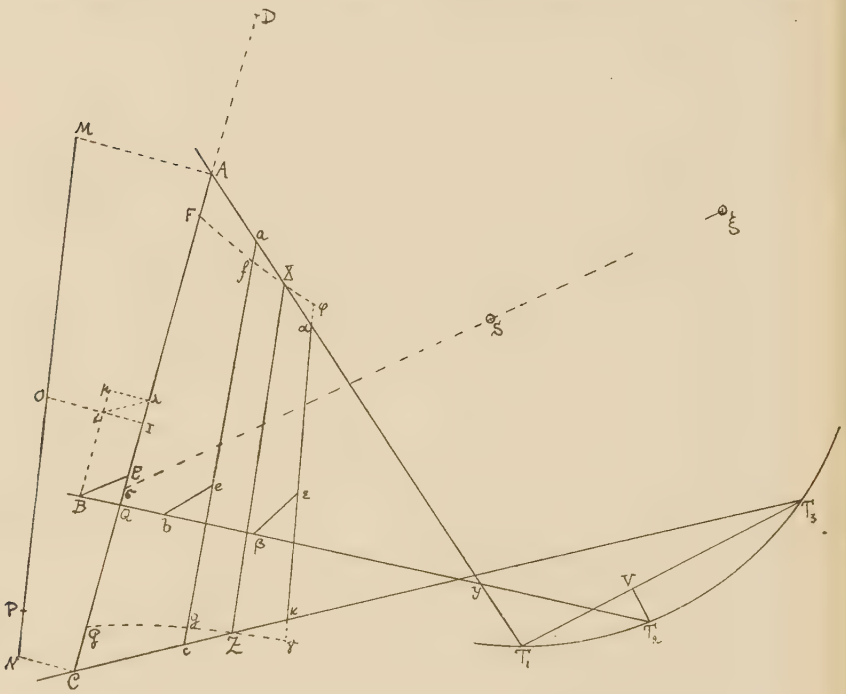

FIG. 4.

$\tau_1 : \tau_2$. The new points A and C will represent more exactly the projections of the comet.

(5) Erect the perpendiculars AM and CN to the line AC and put on them the lengths

$$AM=AT_1 \tan \beta_1;\ CN=CT_3 . \tan \beta_3,$$

where β_1 and β_3 are the latitudes at the first and third observations. Draw the line MN and the perpendicular IO to AC and denote by O the point of intersection of MN with IO ; having constructed anew the parallelogram $i\mathrm{I}\lambda\mu$, lay off on AI produced the length ID=Sμ+$\frac{2}{3}i\lambda$, and having measured OD, compute the length MP by the proportion

$$MP : X=\sqrt{a} : \sqrt{OD}$$

where a is the length of the semi-major axis of the earth's orbit in the scale of the drawing, X having been computed as described above. Put the length MP on the line MN.

If the point P falls on the point N, then A, B, C are the true projections of the comet on the plane of the ecliptic. If the point P does not coincide with N, cut off from AC the length CG=NP so that the point G is on the same side of C as P is of N.

(6) Taking afterwards instead of the point B the points b, β, construct in the manner explained above the corresponding points a, c, g, α, κ, γ.

(7) Through the points G, g, γ draw an arc of a circle and denote by Z its intersection with the line T_3C; this point Z is the required projection of the comet at the time of the third observation.

(8) Draw from the points A, a, α the lengths AF, af, $\alpha\phi$ respectively equal to CG, cg, $\kappa\gamma$, then the points F, f, ϕ will be situated with respect to the line T_1A as the points G, g, γ with respect to T_3C; draw an arc of a circle through these points, its intersection with AT_1 will give the point X, the projection of the comet at the time of the first observation.

(9) Erecting perpendiculars at the points X and Z to the line XZ and cutting from them the lengths

$$A'X=T_1X \,.\, \tan \beta_1 \text{ and } C'Z=T_3Z \,.\, \tan \beta_3,$$

the positions of the comet in its orbit will be obtained, and the orbit itself will be found by constructing a parabola of which the positions of two points and the focus are known.

This description of Newton's construction is given almost in his own words; it is only subdivided into separate steps for the sake of further references.

Newton gives a very concise proof of his construction in the following words: "The proof of this construction follows from the lemmas, because the line AC is divided at the point E in the ratio of the time intervals in accordance with Lemma VII., as it must be according to Lemma VIII.; BE, by Lemma XI., is a part of the line BS or Bξ contained between the arc ABC and the chord AEC and projected on the plane of the ecliptic; MP, by Lemma X., is the length of the chord of the arc described by the comet between the first and the third observation, which chord must be equal to MN if B is the true projection of the comet on the plane of the ecliptic.

"The points B, b, β are not to be taken arbitrarily, but near to their true places. If the angle AQT_2 under which the projection of the chord intersects the line T_2B, is known, the line AC is to be drawn in the necessary direction in such a manner that

$$AC : \tfrac{1}{3}T_1T_3 = SQ^{\frac{1}{2}} : ST_1^{\frac{1}{2}},$$

and having drawn the line SEB, of which the part EB=T_2V, the point B will be obtained from which it is to be drawn."

Then Newton indicates how the next points b, β are to be taken, but it is not necessary for our purpose to enter into these details.

§ 4. The extreme shortness of Newton's explanation renders a commentary necessary in order that it may be clearly understood.

Let accented letters denote the position of that point in the plane of the orbit whose projection is denoted by the same letter unaccented; thus, for instance, B is the projection of the point B′ of the plane of the orbit on the plane of the ecliptic, then we have :—

(1) The length BE is the projection of the length B′E′ on the plane of the ecliptic; this length can be assumed as the height through which the comet falls towards the sun during the first interval of time. The length T_2V being an analogous height for the earth, their ratios, like those of the corresponding accelerations, are inversely as the squares of the distances, hence

$$B'E' : T_2V = ST_2^2 : SB'^2.$$

In order to obtain BE we have to apply the factor $\frac{SB}{SB'}$, hence Newton's expression

$$BE = T_2V \cdot \frac{ST_2^2 \cdot SB}{SB'^3}.$$

(2) Requires no commentary.

(3) and (4) These items are the most important, and the construction used by Newton being only approximate, it is necessary to compare it with the *exact* construction in order to see what has been neglected.

Let it be assumed that the construction is performed in the plane of the orbit, S is the sun, the focus of the parabola, AC a chord, I its middle point, B a point of the orbit, HK a tangent parallel to the chord, hence passing through the vertex μ (fig. 5); in order to obtain this vertex we have to find (by Lemma IX.) on the line HK such a point μ, which being joined with S and with I makes the distances $M\mu = \mu I$. It is obtained by erecting at I the perpendicular Ii, producing it to the point I_1 for which $iI_1 = iI$; the intersection of the straight line SI_1 with HK is the required vertex μ.

The point S being very far from I, which means that the distance SI or SM is very great when compared with $I\mu$, Newton, instead of drawing the line I_1S, draws the line iS, which can be assumed as parallel to I_1S, gets the point of intersection l, and, completing the parallelogram, obtains μ, as stated in (3).

But the tangent HK is not known; we have only the point B, therefore instead of this tangent Newton draws the parallel BG, and hence instead of the point μ he obtains the point μ_1, which is very close to μ.

If extreme accuracy is required, a correction can easily be applied, because the abscissa of the point B is $\mu D = \mu_1 B$, hence

$$\mu_1\mu = BD = \frac{\mu_1 B^2}{4S\mu},$$

but by Lemma IX

$$I\mu = \frac{AI^2}{4S\mu},$$

hence

$$\frac{\mu\mu_1}{I\mu}=\frac{\mu D^2}{AI^2}.$$

By either of these formulæ, having obtained $\mu\mu_1$, we immediately find μ, but Newton prefers to correct directly the place of the point E as explained in his construction, which thus becomes obvious.

In the greater number of cases this correction $\mu\mu_1$ is so small that it can be neglected, and this may be the reason why Newton does not even mention it. But there is a much more important instance, which also he does not mention.

This construction as stated and explained relates *to the plane of the orbit*; Newton performs it for the *projection* on the plane of the

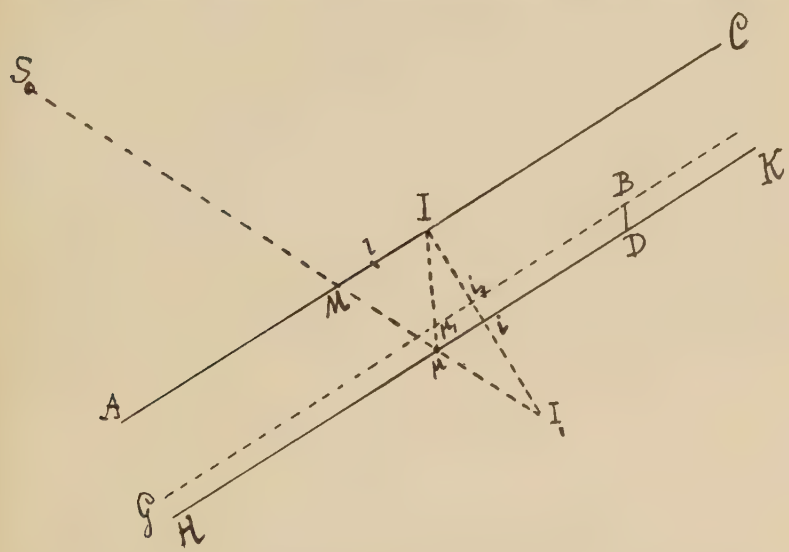

Fig. 5.

ecliptic, but in projection it is no longer applicable and the amount of error thus involved may be too large to be neglected.

When the orbit is projected on the plane of the ecliptic the sun does not remain in the focus of the projected parabola, and thus the focal properties of the point S must not be used.

After projection the point I remains in the middle of A and C, parallel lines remain parallel, but *perpendiculars* in the plane of the orbit are not perpendiculars in projection; equal lines in the orbit, but of different direction, are no longer equal in projection. Hence in projection the line I*i* does not remain perpendicular either to the projected chord or to the tangent, and in projection the line Iμ is not equal to Mμ, hence the point *i*, from which the construction of the parallelogram $i\lambda\mu$I was commenced, is unknown, because the line *i*I is *not* perpendicular to the projected chord AC (fig. 4), and this parallelogram iI$\lambda\mu$ is not a "rectangulum" as Newton calls it.

What is the effect of this slight discrepancy on the final result ?

The position of the point μ is wanted in order to obtain $S\mu$, which enters afterwards into the computation of the length of the chord c (Lemma X.); $M\mu$ or $I\mu$ being small with respect to $S\mu$, the relative error in $S\mu$ will be small and the orbit will be determined with sufficient accuracy to be afterwards corrected by "*calculum arithmeticum*," as Newton says, and hence practically it is of no importance, but if we try to verify the results obtained by Newton's formula (4) by comparing with formula (3) we find a slight difference, although the results ought to be identical. This seems to be the reason why Newton's method was considered only an "approximate" one.

If it is required to make Newton's method precise, the procedure is obvious; we have to make the drawing of the "parallelogram," not in projection but in the plane of the orbit itself, or if in the plane of the ecliptic, by taking not the projection but the figure obtained by rotation of the plane of the orbit about the line of nodes.

Let us consider, *e.g.*, fig. 4. Produce MN till it meets AC produced in the point H. The line SH is the line of nodes; this being determined, we can immediately obtain the positions A′, B′, C′ of the corresponding points after the plane of the orbit is brought to coincidence by rotation about the line of nodes, and hence perform the theoretically exact determination of the vertex μ in complete accordance with Newton's lemmas.

It may be remarked here that when an approximate position μ_1' of the vertex is found, a corrected value of $I'\mu'$ can be calculated in accordance with Newton's Lemma XI. by the formula

$$I'\mu'=M'\mu'=\tfrac{1}{2}\kappa^2\cdot\left(\frac{\tau}{2}\right)^2\cdot\frac{1}{SN'^2},$$

where

$$SN'=S\mu_1'+\tfrac{1}{3}I'\mu_1',$$

hence an obvious method of successive approximation leading very rapidly to a final position as exact as required.

(5) The distance SP′ (Lemma X.), which is necessary for calculating the length of the chord A′C′, is expressed by the formula

$$SP'=\frac{(S\mu'+\tfrac{1}{3}M'\mu')^2}{S\mu'},$$

where accented letters represent points in the plane of the orbit.

Hence we have

$$S\mu'^2=S\mu^2+IO^2,$$

if for sake of simplicity we neglect the difference of elevation of the points μ' and I′ above the plane of the ecliptic.

Then we have

$$i'\lambda'=i\lambda\frac{S\mu'}{S\mu}$$

and

$$SP'=S\mu'\left\{1+\tfrac{1}{3}\frac{i\lambda}{S\mu}\right\}^2=S\mu'+\tfrac{2}{3}i\lambda\frac{S\mu'}{S\mu}+\ldots$$

$$=S\mu'+\tfrac{2}{3}i\lambda+\ldots=\sqrt{(S\mu+\tfrac{2}{3}i\lambda)^2+IO^2}=OD,$$

neglecting such terms as $\frac{i\lambda^2}{S\mu}$ and others of the same order.

(6) No commentary required.

(7) The method here explained is nothing else than what is known as "trial and error." The points F, f, ϕ are the "false positions" of the point X. The "arc of circle" which Newton draws is employed instead of what in graphical interpolation is called a "fair curve."

Once the true positions of the points A and C have been found, the remaining part of the determination of the orbit is too well known to require explanation.

§ 5. Newton considers in the *Principia* as an example the comet of the year 1680. Before making a few remarks about this example, I shall refer to a few examples for recent comets which I have calculated by Newton's method, but with the alterations explained above, having, of course, reduced the graphical process to an analytical one ("*calculus arithmeticus*" mentioned by Newton), the graphical construction being used only to obtain the initial approximation. All the details of these calculations can be found in my "Lectures on the Determination of the Orbits of Comets and Planets from a few Observations," published in the *Transactions of the Naval Academy of St. Petersburg*, 1911, in Russian. As a first example I took the comet 1896, IV., Bauschinger, *Bahnbestim.*, p. 359. The elements calculated by Bauschinger are :—

Longitude of the node	$\Omega=150^\circ\ 33'\ 7''$
Inclination	$i=\ 88^\circ\ 29'\ 28''$
Longitude of perihelion	$\omega=\ 37^\circ\ 46'\ 25''$
Parameter	$\log q=0{\cdot}044192$
Passage of perihelion	$T_0=1896$ July, $9{\cdot}0423$.

The elements obtained on calculating by Newton's method are :—

$$\Omega=150^\circ\ 35'\ 43''{\cdot}7$$
$$i=\ 88^\circ\ 29'\ 10''{\cdot}1$$
$$\omega=\ 38^\circ\ \ 4'\ 43''{\cdot}9$$
$$\log q=0{\cdot}0454748$$
$$T_0=1896 \text{ July, } 9{\cdot}2205.$$

The residual errors for the second observation are :—

(1) By Bauschinger :

$$a_2 \text{ cal.}-a_2 \text{ obs.}=-1''{\cdot}7$$
$$\beta_2 \text{ cal.}-\beta_2 \text{ obs.}=+0''{\cdot}2.$$

(2) By Newton's method :

$$a_2 \text{ cal.}-a_2 \text{ obs.}=-0''{\cdot}7$$
$$\beta_2 \text{ cal.}-\beta_2 \text{ obs.}=-0''{\cdot}4.$$

The chord c when calculated
(a) by Newton's formula is 0·1191758;
(b) by Euler-Lambert's formula is 0·1191762;
the values of the radii being

$$r_1 = 1{\cdot}465534$$
$$r_3 = 1{\cdot}525992,$$

the time interval $\tau = 5{\cdot}99095$ days.

This example shows clearly the equivalence of the two sets of formulæ, and that by using Newton's method results of any desired precision can be obtained.

It is well known that for Olbers' method there is an exceptional case, viz. when the three observed positions of the comet lie on a great circle passing through the place of the sun at the time of the second observation.

For Newton's method this case does not present any exception, but obviously for Newton's method the exceptional case is when the difference of the geocentric longitudes for the first and the third observations is small. Therefore for the next two examples I took one case exceptional for Olbers' method and one exceptional for Newton's.

The elements of comet 1869, III., are calculated in the first volume of Oppolzer's *Lehrbuch zur Bahnbestimmung*, p. 330 (2nd ed.), by a modification of Olbers' method.

The elements found are :—

$$\Omega = 292^\circ\ 56'\ 42''{\cdot}0$$
$$i = 6^\circ\ 56'\ 9''{\cdot}6$$
$$\omega = 107^\circ\ 38'\ 43''{\cdot}2$$
$$\log q = 0{\cdot}042524$$
$$T_0 = 1869\ \text{November},\ 20{\cdot}36747.$$

From the same data, applying Newton's method, I have obtained

$$\Omega = 292^\circ\ 58'\ 11''$$
$$i = 6^\circ\ 56'\ 15''{\cdot}6$$
$$\omega = 107^\circ\ 33'\ 24''{\cdot}2$$
$$\log q = 0{\cdot}042530$$
$$T_0 = 1869\ \text{November},\ 20{\cdot}32284.$$

The residual errors for the second observation are :—
By Oppolzer :

$$\alpha_2\ \text{cal.} - \alpha_2\ \text{obs.} = -\ 4''{\cdot}1$$
$$\beta_2\ \text{cal.} - \beta_2\ \text{obs.} = -34''{\cdot}1.$$

By Newton's method :

$$\alpha_2\ \text{cal.} - \alpha_2\ \text{obs.} = +42''{\cdot}6$$
$$\beta_2\ \text{cal.} - \beta_2\ \text{obs.} = -26''{\cdot}0.$$

Consider now the comet of the year 1781 the elements of which are calculated by Legendre in his treatise, *Nouvelles méthodes pour la détermination des orbites des comètes*, Paris, 1806.

The data for this comet are :—

No.	Time. 1781 Nov. d h m s	Comet. Geoc. Longitude. α. ° ′ ″	Comet. Geoc. Latitude. β. ° ′ ″	The Earth. Hel. Longitude. L. ° ′ ″	The Earth. log R.
1	14 8 29 44	307 14 45	+55 17 9	52 54 2	$\bar{1}$·994864
2	19 8 29 44	306 51 26	39 14 48	57 57 4	·994426
3	29 8 29 44	306 42 20	31 4 52	63 0 41	·994028

Here the geocentric motion in longitude is only 32′, hence the lines T_1A and T_3C (fig. 4) are almost parallel; the method as given in the *Principia* is obviously unsuitable.

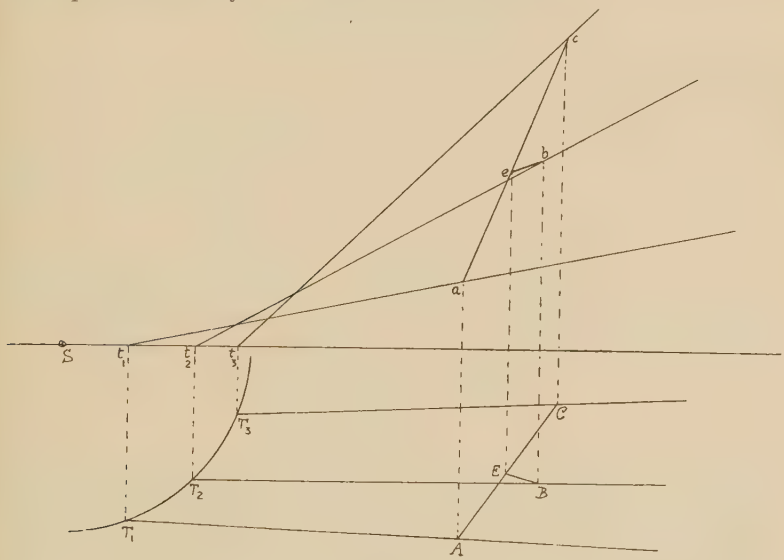

Fig. 6.

But a little consideration shows that if we make the construction not in the projection but in the plane of the orbit itself, we need not bring it into coincidence with the ecliptic, as any other plane would do perfectly well, and this plane is only an auxiliary one.

For instance, we can make our drawing on any plane perpendicular to the plane of the ecliptic. In our case it is convenient to take the plane drawn through the sun at the time of the second observation, and as the second plane of projection we take the ecliptic.

Newton's construction is first performed in the plane perpendicular to the ecliptic, and transferred either to the plane of the ecliptic or to the plane of the orbit. The construction for this case is shown in fig. 6, with an obvious notation corresponding to fig. 4.

The working formulæ for the calculation are quite as easily established for this case as for the general one.

Performing this calculation, I obtained the following elements :—

$$\Omega = 77^\circ\ 55'\ 48''$$
$$i = 27^\circ\ 01'\ 14''$$
$$\omega = 61^\circ\ 57'\ 00'' \qquad (\Omega - \omega) = 15^\circ\ 58'\ 48''$$
$$\log q = \bar{1}\cdot 982784 \qquad q = 0\cdot 961135$$
$$T_0 = 1781 \text{ November, } 29\cdot 56145 = 1781 \text{ November, } 29^d\ 13^h\ 28^m\ 29^s.$$

Legendre gave as final elements :—

$$\Omega = 77^\circ\ 22'\ 55''$$
$$i = 27^\circ\ 12'\ 4''$$
$$\Omega - \omega = 16^\circ\ 3'\ 7''$$
$$q = 0\cdot 960995$$
$$T_0 = \text{November } 29^d\ 12^h\ 42^m.8.$$

The residual errors are :—
By Legendre :

No.	α cal.$-\alpha$ obs.	β cal.$-\beta$ obs.
1	$+194''$	$-115''$
2	$+100$	-55
3	$+28$	$+92$

For Newton's method we need not recalculate the first and third observations, because they are used for the calculation of the elements; for the second we obtain.

$$\alpha_2 \text{ cal.} - \alpha_2 \text{ obs.} = -11''$$
$$\beta_2 \text{ cal.} - \beta_2 \text{ obs.} = -1''.$$

§ 6. The examples given above show that Newton's method, applied not literally but by its true sense, gives results of any desired exactitude, and it is impossible to agree with Bauschinger's words (*Bahnbestimmung*, p. 388): "Newton's method was performed graphically and was applicable. Halley determined a multitude of orbits by this method, and all the methods of the eighteenth century are more or less based on it. *Newton used the property that the chord is subdivided by the mean radius vector proportionally to the time, hence he replaces, as observed by Lambert, the sectors by the corresponding triangles; what his method is wanting in order to render it perfect is Euler-Lambert's theorem, instead of which he was satisfied with an approximate relation between the chord, the radii vectores, and the time.*"

The whole sentence printed in italics is wrong; not only did Newton not use the subdivision of the chord by the mean radius vector but he invented the wonderfully exact construction of Lemma VIII. As to the relation between the chord, the radii vectores, and the time, the relation of Lemma X. is just the so-called Euler-Lambert theorem, which is simply an analytical expression of Newton's Xth lemma. I wish to correct this statement in a standard treatise of great excellence, because it may induce the student not to pay due attention to Newton's method, and to overlook its real meaning and perfection.

§ 7. It is, of course, of interest to examine Newton's own example—the comet of the year 1680.

The data used by Newton are :—

Date.	True Time.	Sun's Longitude.	The Comet's. Longitude.	The Comet's. Latitude.
	h m s	° ′ ″	° ′ ″	° ′ ″
1680 Dec. 21	6 36 59	281 6 44	305 8 12	21 42 13 N.
1681 Jan. 5	6 1 38	296 22 18	8 48 53	26 15 7
1681 Jan. 25	7 58 42	316 45 36	39 35 0	17 56 30

Then with our notation

$$a=10000 \qquad T_2V=455$$
$$ST_2=\ 9842{\cdot}1,$$

and taking first $T_2B=5657$, Newton obtained

$$SB=9747\ ;\ BE=412\ ;\ S\mu=9503\ ;\ i\lambda=413\ ;$$

and by the second operation

$$BE=421\ ;\quad OD=10186\ ;\quad X=8528{\cdot}4.$$
$$MP=8450\ ;\ MN=8475\ ;\quad NP=25.$$

For the second position he took $T_2b=5640$ and obtained

$$T_1X=4775\ ;\ T_3Z=11322.$$

No other values are given but the final elements :—

$$\Omega=271^\circ\ 53'$$
$$i=\ 61^\circ\ 20'\ 20''$$
$$\omega=\ \ 8^\circ\ 38'$$
$$2q=p=236{\cdot}8$$
$$T_0=1680\ \text{December},\ 8^{\text{d}}\ 0^{\text{h}}\ 4^{\text{m}}$$
$$(a=10000).$$

" All these," says Newton, " I have determined graphically, using a scale of equal parts and the chords of the angles taken from a table of natural sines, constructing a sufficiently large drawing in which the semidiameter of the orbit of the earth (10000 parts) was equal to $16\frac{1}{3}$ inches."

It is readily seen that the last figures of such numbers as 4775, 11322, 8475, 10186 would be represented on the drawing by a length of about $\frac{1}{600}$ of an inch, the computation will show that all the numbers above are exact, hence Newton either possessed as a draughtsman the same *vis prope divina*, or he used the *calculus arithmeticus* or some other artifice. It may be observed that such a precision can be obtained if at the same time two drawings be made, the principal one which is described above and the other only an auxiliary one enlarged 50 or 100 times, containing only the points B, E, and the parallelogram $i\lambda\mu$I, or rather B′, E′, $i'\lambda'\mu'$I′, the corresponding points of the plane of the ecliptic. On the first drawing the directions of the chord and of the

other lines are determined graphically, and their lengths computed; these directions are to be transferred to the second drawing on which the " corrections " of the lengths are then determined graphically.

It may be questioned whether Newton's words *construendo schema satis amplum* relate to an analogous artifice by which results of the required precision can be obtained.

Then there arises another difficulty: Newton gives only a single radius vector ST_2 of the earth's orbit. Has he assumed it to be the arc of a circle, or has he taken the values for the ellipse, and what solar tables did he use?

All this makes a verification of his results almost impossible—one has always something doubtful. The nearest results to Newton's I have obtained were when I used Leverrier's tables of the sun, reduced Newton's construction as described in the *Principia* to numerical computation, but corrected the position of the point μ for the distance of the point B from the tangent.

Of course in this case there was a slight difference between the value of the chord calculated by Newton's Xth lemma and by the Eulerian formula. It is obvious from what has been explained above that this difference can only be avoided when we perform the exact construction in the plane of the orbit.

Newton describes a simplified graphical solution which he calls " approximate " (*quam proxime*), but his lemmas contain as complete and exact a solution of this *longe difficillimum* problem as may be desired, and this on seven pages of most sublime geometrical reasoning quite as simple as powerful.

A Hydrogen Spectrum of Constant Frequency-difference.

By J. W. Nicholson, F.R.S.

By a spectrum of constant frequency-difference we mean, of course, one in which the lines fall into groups of two or more which repeat themselves at intervals while preserving the same differences of wave-number between the component members of any group. In spectra of great complexity, the discovery of such groups is the first step towards an orderly arrangement in series, and the first sign that such a series arrangement is possible.

The most classical instance is perhaps argon, where nearly all the lines fall into three groups. By adding quantities A, B, in wave-number, to any line in one set, lines were obtained in the other two sets. The same is true of flame spectra, such as those of lead and tin, which have subsequently been put into series. That the existence of such groups involves the existence of a series arrangement was placed beyond doubt by Paschen's work on neon.

The group-differences are non-recurrent, *i.e.* the lines do not obey a formula like

$$n = A + B\tau$$

as in certain band types of spectra, but merely exist in groups such as, in wave-number,

Group 1. $n_1, n_2, n_3 \ldots$
,, 2. $n_1+A, n_2+A, n_3+A \ldots$
,, 3. $n_1+B, n_2+B, n_3+B \ldots$

and so on.

We have found that hydrogen possesses an arrangement of this kind, which involves many of the strongest lines in its secondary spectrum, and in the present note a short account of it is given. It seems probable that, apart from the Fulcher and other bands discussed in an earlier paper,* the greater number of the strong lines in its spectrum fall into the present scheme, though we now give only a selection of some of the more interesting and conspicuous. Several of these are in fairly close accord with lines of unknown origin occurring in celestial spectra. We may cite, for example, the wave-lengths

λλ4634, 4934, 4580, 4079.

The main fact which seems to render these groupings especially important is that the lines are essentially those which show a normal Zeeman effect—comparison may be made with Dufour's work.† The band-lines already discussed were essentially those which did not show the effect, and there is now a general belief that they arise from hydrogen molecules, and are therefore not to be expected in celestial spectra. But a normal Zeeman effect tends to imply that the lines now to be discussed have an atomic origin—which might be negatively-charged hydrogen atoms. At least it would seem that they come from systems in which one electron is more remote from the nucleus, or nuclei, than any others throughout its orbital motion—with the inevitable consequence, on the principles of the quantum theory, of series arrangements of normal type with Rydberg's constant. They and the band lines must at least come from essentially different systems, and the lines showing Zeeman effect and series relations cannot be dismissed from consideration in the determination of possible astrophysical origins.

They can also be, under certain conditions, to a great extent isolated from the rest of the spectrum. It was, in fact, the study of a peculiar spectrum found by Ames ‡ which originally supplied the clue to their elucidation. The exact variation from the more usual conditions, which was operative in Ames' experiments towards the extreme red end of the spectrum, is very uncertain, but it is sufficient for our purpose to know that effective conditions exist.

Ames only found 61 strong lines. In the region we discuss, in which 50 appear, the wave-number difference 1313 may be shown to occur six times, so that a quarter of the lines at once fall into wide doublets. When this suggestion is tested on the other strong lines in the region λλ3700–4950—which are known to show Zeeman effect in a normal manner—the majority fall into the scheme readily. A selection of the strongest is given in Table I.

* *Monthly Notices*, **85**, 449 (1925).
† *Ann. Chim. Phys.* (8), **9**, 361–432 (1906).
‡ *Phil. Mag.* (30), p. 33 (1890).

TABLE I.

Group I.				Group II.		
λ.	n. = Wave-number.	i.	Δn.	λ.	n.	i.
4966·98	20133·0	3	1312·7	4662·94	21445·7	4
4955·94	20177·8	4	1312·9	4653·18	21490·7	4
4934·43	20265·8	8	1312·0	4634·20	21578·8	9
4876·09	20508·5	4	1312·3	4582·79	21820·8	6
4873·30	20520·4	4	1312·7	4580·19	21833·1	7
4212·67	23737·9	6	1312·2	3992·00	25050·1	4
4079·03	24515·6	8	1312·5	3871·75	25828·1	5
4069·80	24571·2	6	1313·1	3863·35	25884·3	5
3924·57	25480·5	3	1312·9	3732·25	26793·4	3

Here i is the tabular intensity, as given by Watson, and Δn the constant difference of wave-number between the two groups. We have, as stated, only selected the very strong lines in any region. The table is not without certain correspondences with the nebular and the Wolf-Rayet spectra, and may be significant in this respect.

The intensities in the two groups are in fair accord, though a constant ratio of intensities is not to be expected.

Ames' spectrum contains lines without Zeeman effect, but these appear to belong to a band which we shall not now discuss. A spectrum isolated by Frost * is of the same mixed type, and was obtained from a helium tube containing only comparatively small amounts of hydrogen.

We shall finally indicate another large group of lines with Zeeman effect, clearly related to the one above, and with wave-number differences nearly the same. This occurs mainly on the red side of λ4900, but includes apparently all the stronger lines with Zeeman effect in a region where none are, in fact, very conspicuous. It is exhibited in Table II.

TABLE II.

Group (*a*).				Group (*b*).		
λ.	n.	i.	Δn.	λ.	n.	i.
4973·43	20106·85	6	1299·75	4671·46	21406·60	2
4966·98	20132·96	4	1299·72	4665·73	21432·88	2
4969·38	20123·26	3	1299·55	4667·90	21422·91	1
4928·88	20288·60	5	1300·07	4632·06	21588·67	6
4873·20	20520·40	4	1300·37	4582·79	21820·77	6
4856·73	20590·00	4	1299·98	4568·30	21889·98	7
5055·25	19781·40	7	1299·82	4743·56	21081·22	1
5168·99	19346·10	0	1299·58	4843·65	20645·60	1
5266·22	18988·95	4	1299·65	4928·88	20288·60	8
5272·48	18966·40	4	1299·40	4934·43	20265·77	8
5343·43	18714·58	0	1299·03	4996·60	20013·61	2
5642·54	17722·52	4	1300·5	5256·78	19023·05	2
6365·07	15710·75	0	1299·6	5878·73	17010·35	2
6372·41	15692·65	0	1299·95	5884·91	16992·60	2

* *Astrop. J.*, **16**, 100 (1902).

The intensities are again those of Watson,* which are not in close agreement with those of Merton and Barratt † in all cases. The latter are more in accord with the arrangement suggested. We notice that lines beyond λ6000 not belonging to the Fulcher bands come into the scheme.

It should also be added that the lines in these two groupings I. and II., and (*a*) and (*b*)—which naturally have certain members in common, as forming parts of a series arrangement—appear to be the lines most characteristically unaffected by the presence of helium. This fact alone suggests that they are related, and agrees with the observations of Frost,‡ and some later plates taken by Merton and Nicholson, but not published.

Our general conclusion is that the lines of the secondary spectrum of hydrogen are of two main classes. Those which show no Zeeman effect are band-lines of wide separation, of molecular origin. Those which show the effect belong to a series spectrum of some complexity, of a type nearer to that of neon than to that of helium, and quite possibly of an atomic origin and with some degree of representation in celestial spectra.

On a Method for deducing approximately the Focal Lengths and Magnifying Powers of Early Telescopes. By S. B. Gaythorpe.

Of the early telescopes little is known. Very few of those primitive instruments have been preserved, and contemporary accounts record little more than the magnifying power, with here and there a rough estimate of the radii of curvature of the lenses. If, however, a telescope has been applied to the observation of the sun by projection, and measurements are available, it will nearly always be possible to deduce approximately the magnifying power, and sometimes also the focal lengths of the lenses.

In the present paper it is proposed to develop this method of inquiry, and illustrate it by application to a telescope of the Galilean type, used (1640–41) by William Gascoigne, the inventor of the micrometer.

If β denote the linear semi-diameter of an object, and β' that of the final image formed by a system of two lenses, then it can be shown that

$$\beta' = \beta \frac{x'y}{xy'} \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where x, y are the distances of the object and the final image from the outer principal points of the first and second lenses respectively, and x', y' the foci conjugate to those respective distances, measured from the inner principal points in the usual way. The distance, c, between the two lenses will accordingly be $x'+y'$.

* *Proc. Roy. Soc.*, p. 190 (1909).
† Bakerian Lecture, *Phil. Trans. Roy. Soc.*, 1923. ‡ *Loc. cit.*

Now, in the case of the sun, x' becomes the principal focal length, f, of the first lens, or object-glass, while β/x is the tangent of the sun's apparent semi-diameter, α. Hence

$$\beta'=\frac{fy\tan\alpha}{y'} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Further, it can be shown that if f' denote the principal focal length of the second, or eye-, lens,

$$\frac{1}{f'}=\frac{1}{y}+\frac{1}{y'} \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

Hence, if both lenses are collective, as in the Keplerian telescope, y', and therefore β', are both positive, and the final image is erect; but if the second lens is dispersive, as in the Galilean telescope, f', y', and β' are all negative, and the image is inverted.

If the diameter of the solar image and its distance from the eye-lens are known, as also the distance between the two lenses, their principal focal lengths can be found from equations (ii) and (iii) and the relation $c=x'+y'$, as follows :—

$$y'\beta'=\beta' c-\beta' f=fy\tan\alpha\,;\ \text{hence}\ f(\beta'+y\tan\alpha)=\beta' c,\ \text{or}$$

$$f=\frac{c}{1+\dfrac{y\tan\alpha}{\beta'}} \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

In the case of the Keplerian telescope, the denominator is greater, and in the case of the Galilean, less than unity. Hence, in the former, the focal length of the object-glass is less, while, in the latter, it is greater than the distance between the lenses.

Lastly,

$$y'=c-f=\frac{fy\tan\alpha}{\beta'} \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

and

$$f'=\frac{yy'}{y+y'} \quad . \quad . \quad . \quad . \quad . \quad \text{(vi)}$$

When the data are insufficient for a solution of the above equations, an approximate value of the magnifying power may still be found. Substituting for y' in equation (ii) its equivalent $\frac{f'y}{y-f'}$ we have $\beta'=\frac{fy-ff'}{f'}\tan\alpha$. But f/f' is the magnifying power, m. Hence

$$\beta'=(my-f)\tan\alpha \quad . \quad . \quad . \quad . \quad \text{(vii)}$$

In the case of the Galilean telescope, m, being the ratio of a positive and a negative quantity, must be regarded as negative; and so not only is the image inverted, as we have already seen, but also for an equal distance, y, from the eye-lens, its semi-diameter is larger by $2f\tan\alpha$ than that of the corresponding image formed by a Keplerian telescope

with lenses of the same focal lengths, and consequently of equal magnifying power. As, in addition to the larger image, the total length of the combination is shorter in the case of the former by nearly twice the focal length of the eye-lens, it is evident that in the early days of the telescope, and for the purpose under consideration, the Galilean type possessed advantages over the Keplerian, other than the superior distinctness of the image mentioned by Professor Grant (*Hist. of Phys. Astronomy*, p. 525).

In his second letter to Oughtred, not dated, but probably written in 1641 February, printed in Rigaud's *Corresp. of Sci. Men*, 1, 35–59, Gascoigne gives the following measurements of two "London best sale glasses," which he used as a Galilean telescope :—

Object-glass (*plano-convex*).

Diameter	1·2 inches
Thickness	0·105 ,,
Radius of Curvature	25·8 ,,
Refractive Index	sin 2° 5′ 52″/sin 1° 19′ 56″.

This makes $\mu=1{\cdot}5744$, whence $f=r/\mu-1=44{\cdot}91$ inches.

Eye-lens (*plano-concave*).

Diameter	0·75 inches
Thickness	0·055 ,,
Radius of curvature	2·17 ,,
Refractive Index (first determination)	sin 15°·40/sin 9°·95.
,, ,, (second ,,)	sin 15°·2449/sin 9°·8047.

The first of these values gives $\mu=1{\cdot}5369$, and the second, 1·5441. The mean is 1·5405, whence $f'=-4{\cdot}015$ inches, and $m=f/f'=11{\cdot}19$.

The same letter also contains the following measurements of the semi-diameter of the solar image, its distance from the concave surface of the eye-lens, and the distance apart of the two lenses, in three observations, dated 1640 October 31, December 18, and 1640-41 January 20, O.S.

Obs.	Semi-diam. of Image.	Distance from Eye-lens.	Distance between Lenses.	⊙'s Apparent Semi-diam.
	Inches.	Inches.	Inches.	′ ″
1	6·00	110·21	40·91	16 12·3
2	2·50	43·07	40·97	16 17·8
3	6·00	109·46	40·85	16 14·7

The figures in the last column have been computed for the given dates from the *Nautical Almanac* value of the sun's mean apparent semi-diameter and approximate values of the orbital elements. It would, however, have been sufficient to take the figures directly from the *Nautical Almanac* for the dates most nearly corresponding to those of the observations.

It should be mentioned that the lenses were so arranged that their

plane surfaces were directed towards the object. As the distances in the last column but one were measured from the convex surface of the object-glass to the plane surface of the eye-lens, they require to be increased by 0·036, or the thickness of the latter divided by its refractive index.

This adjustment having been made, a solution of the equations gives the following results :—

Obs. 1.	f=44·83 inches,	f'=−4·02 inches.
„ 2.	44·65 „	3·98 „
„ 3.	44·74 „	4·00 „

The mean values are f=44·74 and f'=−4·00, whence m=11·185. The refractive indices corresponding to these values of f and f', and the radii of curvature already given, are 1·5767 and 1·5425 for the object-glass and eye-lens respectively, differing from Gascoigne's independent determinations by quantities of the order of 0·002.

As a further illustration of the method, let us take the telescope specified by Flamsteed in his first communication to the Royal Society, dated 1669 November 24 (Rigaud's *Corresp. of Sci. Men* **2**, 80), where he advises the observers of solar eclipses "to cast the species of the sun through a telescope of at least two feet, free from spots and colours, on an extended paper placed at least six feet behind the eye-glass, or so far as that his species may appear at least six inches over. . . ."

There is nothing to show whether the telescope was to be of the Keplerian or Galilean type. We have, therefore, β'=+3 or−3, y=72, and α (mean) 16′ 1″·18. Taking an approximate value for f, we find from equation (vii) that the magnifying power would be about 9·2 for a Keplerian and 8·6 for a Galilean telescope. Further, we find that f would be between 21 and 22 in the case of the former, and about 27 in the case of the latter. As with a focal length of only 21 or 22 a power as high as 9 seems unlikely, it is probable that the telescope Flamsteed had in mind was of the Galilean type. On this assumption, taking c=24, we obtain as the foci of the two lenses 27·0 and −3·15 respectively, and 8·57 as the magnifying power.

Observations of Novæ, 1924. By W. H. Steavenson.

The photometric observations of Novæ which follow are in continuation of those made and published in previous years, the same comparison stars having been used in each case. Nearly all the observations were made with powers of 120 or 280 on the 6-inch Wray refractor.

Nova Cassiopeiæ (1572).

The field of this Nova was examined on 11 nights during the year, but no changes were noted in the magnitudes of the small stars

within a few minutes of arc of the approximate position known to have been occupied by Tycho's star. In particular, the two brightest stars near to the computed place were carefully watched. They are both of about the twelfth magnitude. The more northerly and following of the two lies the nearer to this place, and has been thought by more than one observer to be probably identical with the great Nova. It is slightly brighter than the other star, and was suspected of variability by Hind and D'Arrest. It seems possible that such suspicions may have been due to physiological causes, for the star is distinctly reddish in colour, and, like all stars of this tint, affects the eye differently according to the state of illumination of the field at the time of observation. Thus, in twilight and moonlight, it appears unduly bright; but, when care has been taken always to observe it in a dark sky, there appears to have been no certain variation in its light during the past three years. In 1922 the late Dr. Rambaut very kindly sent me a photograph of the region, taken at my request with the 24-inch telescope of the Radcliffe Observatory on September 15 of that year. This photograph confirms my visual estimate by showing that the small star already referred to has quite a large positive colour-index. This fact would seem to be inconsistent with the supposition that it is actually Tycho's star, unless the Wolf-Rayet type of spectrum, which we have come to associate with the final stages of a Nova, is but a short-lived phase.

Nova Ophiuchi (1848).

Observed on 23 nights between May 1 and September 23. The observations show a continuation of the variability noted in previous years, and there is again a suggestion of a period of about 40 days; but the curve for 1924 is too much broken by bad weather to yield a definite result in this respect.

Date.	Mag.	Date.	Mag.	Date.	Mag.
1924 May 1	12·52	1924 June 23	12·4	1924 July 28	12·35
7	12·45	26	12·45	30	12·32
26	12·35	July 3	12·35	31	12·35
27	12·35	4	12·3±	Aug. 2	12·42
29	12·35	6	12·2	5	12·35
June 5	12·3±	10	12·1	19	12·12
19	12·55	21	12·45	Sept. 23	12·4±
20	12·6	24	12·45		

Nova Cygni (1876).

Observed on 8 nights between July 21 and October 23. It is only on the clearest nights that this Nova can be seen at all with the 6-inch telescope, and the small differences noted can hardly be relied upon as sound evidence of variation. They do, however, show that the changes, if any, were not large.

Date.	Mag.	Date.	Mag.
1924 July 21	14·81	1924 Aug. 2	14·76
24	14·59	5	14·62
30	14·81	19	14·71
31	14·81	Oct. 23	14·7±

Nova Lacertæ (1910).

Observed on 3 nights. This Nova is watched less closely than some of the others, as it has shown little or no tendency to vary in its light since it ceased to fade. It has now for some years been of about the same brightness as before its outburst.

Date.	Mag.
1924 July 24	14·1
Aug. 5	14·05
19	14·1

Nova Aquilæ (1918).

Observed on 37 nights between May 5 and October 22. The magnitude has been pretty steady at about 10·35, but there have been slight fluctuations which I am confident are real. However, the entire range in visual magnitude, as observed in 1924, did not exceed $0^{m}\cdot25$. The mean brightness is now practically the same as before the outburst of the Nova.

Date.	Mag.	Date.	Mag.	Date.	Mag.
1924 May 5	10·3	1924 June 24	10·4	1924 July 19	10·45
7	10·35	26	10·5	21	10·3
16	10·35	30	10·3	22	10·3
26	10·35	July 3	10·4	24	10·35
27	10 3	4	10·3	28	10·3
28	10·3	6	10·3	29	10·3
29	10·25	7	10·35	Aug. 5	10·3
June 5	10·35	9	10·35	7	10·35
13	10·35	10	10·4	9	10·35
19	10·35	13	10·4	19	10·4
20	10·4	14	10·4	Sept. 2	10·3
22	10·4	15	10·5	Oct. 22	10·4
23	10·4				

Nova Cygni (1920).

Observed on 31 nights, between April 7 and October 23. This object is still slowly fading. The fall is somewhat irregular, but there is nothing to indicate any definite period in the fluctuations observed. The magnitude of the Nova before the outburst is known to have been less than 15, and it is therefore probable that the brightness will continue to decrease for some time.

Date.	Mag.	Date.	Mag.	Date.	Mag.
1924 Apr. 7	11·7	1924 June 13	12·0	1924 July 6	12·4
8	12·0	19	12·1	7	12·4
9	11·8	20	12·1	9	12·5
May 5	11·9	22	12·2	10	12·6
7	12·0	23	12·2	13	12·6
16	12·05	24	12·25	21	12·0
26	12·05	26	12·5	24	12·4
27	12·05	30	12·35	Aug. 19	12·75
28	12·1	July 3	12·2	Sept. 22	12·75
29	12·05	4	12·25	Oct. 23	12·75
June 5	12·0				

Observations of Nova Persei (1901) *in* 1924–1925.

By W. H. Steavenson.

Owing to unfavourable weather this Nova was not observed on more than 55 nights during the period 1924–25. The light-curve is, therefore, much less complete than those of recent years. It is, however, possible to obtain from it a general idea of the behaviour of the Nova during the period covered by the observations. Perhaps the most striking feature has been the general faintness of the Nova. After 1924 July 21 the magnitude only once rose above 13·0, and during the greater part of the period it lay between 13·4 and 13·8. Not since 1918–19 has the average brightness of the Nova been so low as it was in 1924–25. There have, as usual, been marked fluctuations, but these have been more rapid and less regular than during the previous three years.

Most of the estimates given below were made with the 6-inch refractor, but a few were made with Mr. B. M. Peek's 12¼-inch reflector; for the observation of August 31 the 28-inch refractor of the Royal Observatory was used.

Date.	Mag.	Date.	Mag.	Date.	Mag.
1924 July 15	12·8±	1924 Aug. 19	13·87	1924 Oct. 8	12·96
18	12·74	22	13·57	11	13·40
19	12·5±	25	13·6±	19	13·72
21	12·43	27	13·6±	22	13·72
24	13·27	30	13·52	23	13·23
28	13·85	31	13·23	24	13·47
30	13·8±	Sept. 2	13·42	28	13·70
31	13·72	5	13·28	30	13·50
Aug. 2	13·65	22	13·82	Nov. 3	13·18
5	13·86	23	13·70	7	13·5±
18	13·6±	24	13·77	30	13·67

Date.	Mag.	Date.	Mag.	Date.	Mag.
1924 Dec. 14	13·72	1925 Jan. 23	13·67	1925 Feb. 22	13·67
18	13·77	31	13·42	23	13·62
28	13·70	Feb. 6	13·77	24	13·67
30	13·70	12	13·18	27	13·77
31	13·72	14	13·18	Mar. 15	13·77
1925 Jan. 4	13·77	16	13·42	18	13·57
5	13·79	18	13·28	23	13·6±
6	13·8±				

The small wisp of nebulosity, in contact with the Nova on its south preceding side, is still clearly visible in large apertures. As seen at Greenwich on 1924 August 31, it had the same appearance as when I first saw it on 1916 September 4. The image of the Nova itself is perfectly stellar in character.

Photometric Observations of Novæ, 1924–25. By B. M. Peek.

Nova Persei, 1901.—In spite of the abnormal amount of cloud prevalent during the past apparition, the Nova was observed on 129 nights. The magnitudes of the comparison stars, as also in 1923–24, are those of Seares as used by Barnard and Steavenson. The majority of the observations have been made with the 12¼-inch reflector.

Apart from the somewhat abnormally bright maximum of July 11–12 followed by a secondary about July 21, the magnitude of the Nova for the nine months covered by these observations has been considerably below its normal mean value.

Table I. gives the individual estimates.

TABLE I.

Date. 1924	Mag.	Date. 1924	Mag.	Date. 1924	Mag.
July 4·62	13·04	July 30·49	13·6±	Aug. 20·44	13·62
11·54	12·10	31·50	13·72	21·52	13·72
12·58	12·07±	Aug. 2·50	13·60	24·42	13·72
13·57	12·48	8·52	13·56	27·36	13·62±
14·59	13·05	9·55	13·77	31·52	13·08
16·58	12·78	10·53	13·67±	Sept. 5·46	13·36
18·62	12·55±	11·48	13·67±	7·40	13·61
19·52	12·59	12·54	13·77	9·44	13·52
21·53	12·42	13·53	13·74±	10·42	13·50±
22·52	12·8±	17·44	13·79±	14·45	13·84±
24·50	13·28	18·52	13·60	20·48	13·67±
28·49	13·75	19·45	13·77	21·49	13·53

TABLE I.—*continued.*

Date.	Mag.	Date.	Mag.	Date.	Mag.
1924		1924		1925	
				New time.	
Sept. 22·43	13·60	Nov. 27·41	13·77	Jan. 23·79	13·77
23·44	13·57	29·35	13·80	28·85	13·35
27·46	13·08	30·33	13·80	Feb. 1·77	13·59
28·33	13·32	Dec. 2·25	13·86	2·77	13·67
Oct. 1·41	13·57	3·41	13·82	6·79	13·72±
3·47	13·72	6·38	13·75	12·79	13·15
4·38	13·70	7·24	13·72±	15·79	13·23
5·31	13·35	9·50	<13·2	17·83	13·30
8·30	13·35±	14·34	13·74	18·81	13·11
12·46	13·37±	16·33	13·74	21·80	13·45±
13·38	13·03	17·27	13·85	22·83	13·58
14·29	13·42	18·50	13·82	23·83	13·67
18·43	<13·47	20·26	13·77	24·81	13·54
19·45	13·62	22·38	13·62±	27·83	13·57
20·35	13·63	24·25	13·74	28·80	13·55
23·38*	13·29	25·25	13·60±	Mar. 1·82	13·61
24·29	13·52	26·31	13·57	2·85	<13·3 / >13·7
27·50	13·67±	27·49	13·64	3·88	<13·2
28·49	13·72	28·25	13·75	8·82	13·57
29·53	13·55	28·43	13·59	9·81	13·61
30·29 †	13·47	30·34	13·77	10·81	<13·2
30·50	13·72	30·52	13·67	11·81	13·39
Nov. 2·39	13·57±	1925		12·81	13·42
3·38	13·28	New time.		15·88	<13·2
4·35	13·62	Jan. 4·76	13·77	18·85	<13·2
4·41	13·72	5·82	13·79	21·85	13·55±
5·42	13·77	6·81	<13·5	22·86	13·54
8·27	<13·2	9·77	13·67±	23·84	13·72
10·48	<13·67	10·74	<13·2	24·85	13·47
14·34	13·80	11·88	<13·6	26·84	13·57±
15·44	<13·5	15·79	13·37	27·84	13·57
16·29	13·80	15·86	13·28	Apr. 3·85	13·57
17·36	13·82	17·95	13·5±		
24·31	13·81	18·74	13·77		

From March 15 onwards the full aperture of the mirror was not effective owing to lack of sky-room, so that estimates after March 12 may not carry quite so much weight.

* Weighted mean for night. The record shows 7 hrs. 13·40, 8½ hrs. 13·30, 10¾ hrs. 13·23. There was haze at the first of these observations and the Nova was getting higher in the sky, but probably a real rise in magnitude is indicated.

† Some haze at second observation, but probably a real fall.

Nova Ophiuchi, 1848.—Observed on 25 nights between May 25 and September 14.

The comparison stars are those given by Barnard in *M.N.*, **81**, 167.

Table II. gives the individual estimates.

TABLE II.

Date. 1924	Mag.	Date. 1924	Mag.	Date. 1924	Mag.
May 25	12·5	June 24	12·4	July 28	12·4
27	12·5	25	12·4	30	12·3
28	<12·3	July 4	12·3±	31	12·35
June 2	12·45	11	12·05±	Aug. 2	12·4
8	12·3	18	12·15±	18	12·2
19	12·45	19	12·4	24	12·1
20	12·5	22	12·55±	27	12·0
21	12·55±	24	12·45	Sept. 14	12·2±
23	12·5				

Nova Cygni, 1876.—Observed on 35 nights between June 8 and Dec. 28. The magnitudes of the comparisons stars are those given by Seares as used by Barnard and Steavenson. The difficulties of observation are such as to give little weight to small recorded fluctuations, but it seems probable that there was a rise to a maximum early in July, and that the Nova was relatively bright on September 23.

Table III. gives the estimates.

TABLE III.

Date. 1924	Mag.	Date. 1924	Mag.	Date. 1924	Mag.
June 8	<14·7	July 28	14·9±	Sept. 5	14·78
20	14·8±	30	14·81	23	14·24
21	14·92	31	14·91	28	14·71
23	14·61	Aug. 2	14·81	Oct. 1	14·61
24	14·66	8	14·7±	4	14·81
25	14·61	9	14·71	14	<14·6
July 4	14·40	14	<14·7	19	14·81
11	14·8±	18	14·66±	23	14·91
13	14·9±	19	14·76	Nov. 3	<14·5
19	14·81	20	14·78	16	<14·71
22	<14·7	21	14·71	Dec. 28	14·76±
24	14·91	24	14·66		

Erratum.

P. 478, *for* $(\alpha+3{\cdot}66=247^\circ)$: δ *read* $(\alpha+3{\cdot}66\delta=247^\circ)$:

Addendum.

M.N., **85**, 480, line 13. In connection with my statement on orbits with very high eccentricities, Dr. R. G. Aitken has kindly pointed out that the only decidedly higher eccentricities are 0·96 deduced for Σ1865 by Aitken and Hertzsprung, and for OΣ341 by the former.

Geo. Van Biesbroeck.